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Summary

(M only)

Basis vectors $\{e_\mu\}$

Basis forms $\{\theta^\mu\}$

Dual bases, $\theta^\mu(e_\nu) = \delta^\mu_\nu$

Structure consts, $[e_\mu, e_\nu] = c^\sigma_{\mu\nu}\, e_\sigma$

or $d\theta^\mu = -\frac{1}{2} c^\mu_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta$

Comma notation: $e_\mu f \equiv f_{,\mu}$ for $f: M \to \mathbb{R}$

Now (M, ∇) (but no g, no assumptions on ∇)

$$\nabla_\mu \equiv \nabla_{e_\mu}$$

$$\nabla_\mu e_\nu = \Gamma^\alpha_{\mu\nu}\, e_\alpha \qquad (\text{defn of } \Gamma^\mu_{\nu\alpha})$$

$$\nabla_\mu \theta^\alpha = -\Gamma^\alpha_{\mu\nu}\, \theta^\nu \qquad (\text{from } \theta^\mu(e_\nu) = \delta^\mu_\nu)$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$T(e_\alpha, e_\beta) = T^\mu_{\alpha\beta}\, e_\mu \qquad (\text{defn of } T^\mu_{\alpha\beta})$$

$$T^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha} - c^\mu_{\alpha\beta}$$

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$$R(e_\alpha, e_\beta)\, e_\nu = R^\mu{}_{\nu\alpha\beta}\, e_\mu \qquad (\text{defn of } R^\mu{}_{\nu\alpha\beta})$$

$$R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu_{\beta\nu,\alpha} - \Gamma^\mu_{\alpha\nu,\beta} + \Gamma^\sigma_{\beta\nu}\Gamma^\mu_{\alpha\sigma} - \Gamma^\sigma_{\alpha\nu}\Gamma^\mu_{\beta\sigma} - c^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\nu}$$

0.5

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Cartan's definitions

$$\omega^{\mu}{}_{\cdot\nu} = \Gamma^{\mu}_{\alpha\nu}\,\theta^{\alpha}$$ (Lie-alg.-valued 1-form)

$$T^{\mu} = \frac{1}{2} T^{\mu}_{\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta}$$ (vector-valued 2-form)

$$R^{\mu}{}_{\cdot\nu} = \frac{1}{2} R^{\mu}{}_{\cdot\nu\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta}$$ (Lie alg-valued 2-form.

$$\boxed{d\theta^{\mu} + \omega^{\mu}{}_{\cdot\nu}\wedge\theta^{\nu} = T^{\mu}}$$ 1st Cartan structure.

The LHS can be interpreted as a "covariant exterior derivative". This eqn. can be interpreted as an alternative definition of the torsion (equivalent to the usual one by the calculation above).

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$$= \frac{1}{2}\left(\Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\sigma\nu}\, C^{\sigma}_{\alpha\beta} \right) \theta^{\alpha} \wedge \theta^{\beta}$$

$$= \frac{1}{2}\left(R^{\mu}{}_{\cdot\nu\alpha\beta} - \Gamma^{\sigma}_{\beta\nu}\Gamma^{\mu}_{\alpha\sigma} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\mu}_{\beta\sigma} \right) \theta^{\alpha} \wedge \theta^{\beta}$$

$$= R^{\mu}{}_{\cdot\nu} - \frac{1}{2}\, \omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu} + \frac{1}{2}\, \omega^{\sigma}{}_{\cdot\nu} \wedge \omega^{\mu}{}_{\cdot\sigma},$$

equal

or,

$$\boxed{d\omega^{\mu}{}_{\cdot\nu} + \omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu} = R^{\mu}{}_{\cdot\nu}}$$ 2nd Cartan structure eqn.

Again, take

$$T^{\mu} = d\theta^{\mu} + \omega^{\mu}{}_{\cdot\alpha} \wedge \theta^{\alpha}, \quad \text{apply } d,$$

$$dT^{\mu} = 0 + \left(R^{\mu}{}_{\cdot\alpha} - \cancel{\omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\alpha}} \right) \wedge \theta^{\alpha}$$

$$- \omega^{\mu}{}_{\cdot\alpha} \wedge d\theta^{\alpha}$$

$$d\theta^{\alpha} \longrightarrow T^{\alpha} - \cancel{\omega^{\alpha}{}_{\beta} \wedge \theta^{\beta}}$$

$$\boxed{dT^{\mu} + \omega^{\mu}{}_{\cdot\alpha} \wedge T^{\alpha} = R^{\mu}{}_{\cdot\alpha} \wedge \theta^{\alpha}}$$ 1st Bianchi identity, generalized to case $T \neq 0$.

Finally, take 2nd Cartan structure, apply d:

$$d R^{\mu}{}_{\cdot\nu} = d\omega^{\mu}{}_{\cdot\sigma} \wedge \omega^{\sigma}{}_{\cdot\nu} - \omega^{\mu}{}_{\cdot\sigma} \wedge d\omega^{\sigma}{}_{\cdot\nu}$$

$$= \left(R^{\mu}{}_{\cdot\sigma} - \cancel{\omega^{\mu}{}_{\cdot\alpha} \wedge \omega^{\alpha}{}_{\cdot\sigma}} \right) \wedge \omega^{\sigma}{}_{\cdot\nu}$$

$$- \omega^{\mu}{}_{\cdot\sigma} \wedge \left(R^{\sigma}{}_{\cdot\nu} - \cancel{\omega^{\sigma}{}_{\cdot\alpha} \wedge \omega^{\alpha}{}_{\cdot\nu}} \right)$$

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One might say that the covariant exterior derivative of the curvature 2-form is 0, that this form is closed in this sense.

~~$dR^\mu \wedge \ldots$~~

$$\boxed{dR^\mu_{\cdot\nu} + \omega^\mu_{\cdot\sigma}\wedge R^\sigma_{\cdot\nu} - R^\mu_{\cdot\sigma}\wedge\omega^\sigma_{\cdot\nu} = 0}$$

2nd Bianchi, generalized.

When $T=0$, these eqns should reduce to the previous versions of the Bianchi identities. For the 1st Bianchi ident., this gives

$$0 = R^\mu_{\cdot\alpha}\wedge\theta^\alpha = \frac{1}{2} R^\mu_{\cdot\nu\alpha\beta}\,\theta^\nu\wedge\theta^\alpha\wedge\theta^\beta$$

$$\Rightarrow R^\mu_{\cdot[\nu\alpha\beta]} = 0. \qquad \text{checks.}$$

For the 2nd Bianchi ident., notice that it doesn't involve T at all. But if you want to show equivalence to $R^\mu_{\cdot\nu[\alpha\beta;\gamma]} = 0$, you must use $T=0$.

Now consider the case that we have a metric g and a metric connection $\nabla g = 0$.

Then it is convenient to assume the basis $\{e_\alpha\}$ is <u>orthonormal</u>, i.e.,

$$g_{\alpha\beta} = g(e_\alpha, e_\beta) = \eta_{\alpha\beta} \qquad (\text{pseudo-Riem. case, or } \delta_{\alpha\beta}, \text{ Riem. case}).$$

$$= \text{const. metric of special relativity.}$$

We know that if the curvature tensor $\neq 0$, then there is no <u>coordinate basis</u> such that $g_{\alpha\beta} = \eta_{\alpha\beta}$. But there are always <u>non-coordinate</u> bases that make this true. This is a special kind of vielbein.

There are some special properties of Γ, R in orthonormal vielbeins. First, $\nabla g = 0$ implies

$$0 = g_{\mu\nu,\alpha} - \Gamma^{\beta}_{\alpha\mu}\, g_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu}\, g_{\mu\beta}.$$

Define $\Gamma_{\alpha\mu\nu} = g_{\alpha\beta}\Gamma^{\beta}_{\ \ \mu\nu}$. Note, this $\Gamma_{\alpha\mu\nu}$ is the 1-form index.

Also, in an orthonormal vielbein, $g_{\mu\nu} = \eta_{\mu\nu}$ so $g_{\mu\nu,\alpha} = 0$. Thus,

$$\Gamma_{\mu\alpha\nu} + \Gamma_{\nu\alpha\mu} = 0,$$

and $\Gamma_{\mu\nu}$ is antisymmetric in $\mu\nu$. (Recall in coord. basis w. LC connection, $\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha\nu\mu}$.) This property depends only on $\nabla g = 0$ (the parallel transport proceed by orthogonal (or Lorentz) transformations), it does not require the LC connection.

In terms of Cartan's forms, this condition is

$$\omega_{\mu\nu} = -\,\omega_{\nu\mu} \qquad \left(\omega_{\mu\nu} = \eta_{\mu\alpha}\,\omega^{\alpha}_{\ \cdot\,\nu}\right).$$

Similarly, we have

$$R_{\mu\nu} = -R_{\nu\mu} \qquad \left(R_{\mu\nu} = \eta_{\mu\alpha}\, R^{\alpha}_{\ \cdot\,\nu},\ \text{Riemann-Cartan tensor}\right)$$

for the same reason.

Also note, if in addition $T=0$ (L.C. connection) then
$$\Gamma^{\alpha}_{\ \mu\nu} = -\tfrac{1}{2}\left(C^{\alpha}_{\ \mu\nu} + C_{\mu\ \cdot\,\nu}^{\ \alpha} + C_{\nu\ \cdot\,\mu}^{\ \alpha}\right)$$

Now we consider a change of basis for an orthonormal vielbein. To be specific, we'll assume the pseudo-Riemannian (1+3) case, with $g_{\mu\nu} = \eta_{\mu\nu}$. A change of basis maps one orthonormal vielbein to another.

We are assuming that $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$.

~~Let $e'_\mu = \Lambda^{\alpha}_{\ \cdot\,\mu}\, e_\alpha$, and demand that $g(e'_\mu, e'_\nu) = \eta_{\mu\nu}$, so the new vielbein is also orthonormal.~~

Let $e'_\alpha = \Lambda_\alpha^{\cdot\beta} e_\beta$, defines $\Lambda_\alpha^{\cdot\beta}$. Then demand $g(e'_\alpha, e'_\beta) = \eta_{\alpha\beta}$, and you find

$$\Lambda^\alpha_{\cdot\mu}\, \eta_{\alpha\beta}\, \Lambda^\beta_{\cdot\nu} = \eta_{\mu\nu}$$

where indices are raised + lowered with η. Thus $\Lambda^\alpha_{\cdot\mu}(x)$ is an <u>x-dependent Lorentz transformation</u>. <u>These are gauge transformations in GR.</u> How other things transform:

$$\theta'^\mu = \Lambda^\mu_{\cdot\alpha}\, \theta^\alpha .$$

Any tensor transforms pointwise-linearly in $\Lambda(x)$, for example, the Riemann-Cartan 2-form,

$$R'^\mu_{\cdot\nu} = \Lambda^\mu_{\cdot\alpha}\, \Lambda_\nu^{\cdot\beta}\, R^\alpha_{\cdot\beta} .$$

But the Cartan-Connection 1-form has a less simple transformation law (since Γ is not a tensor):

$$\omega'^\sigma_{\cdot\nu} = \Lambda^\sigma_{\cdot\gamma}\, \Lambda_\nu^{\cdot\beta}\, \omega^\gamma_{\cdot\beta} - \Lambda^\sigma_{\cdot\gamma,\alpha}\, (\Lambda^{-1})^\gamma_{\cdot\nu}\, \theta^\alpha .$$

The extra term on the right is characteristic of the transformation laws for gauge potentials.

$$\boxed{\omega^\gamma_{\cdot\beta} = \Lambda_\sigma^{\cdot\gamma}\, \omega'^\sigma_{\cdot\nu}\, \Lambda^\nu_{\cdot\beta} + \Lambda_\sigma^{\cdot\gamma}\, \Lambda^\sigma_{\cdot\beta,\alpha}\, \theta^\alpha}$$

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in (3+1)-dim. space-time.

Now we deal with the variational formulation of GR. We work in coordinates x^μ. We start with the vacuum (matter-free) case, for which the field eqns are

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0.$$

We seek a Lagrangian density $\mathcal{L}_G$ such that these eqns follow from

$$\delta \int d^4x \sqrt{-g}\; \mathcal{L}_G = 0.$$

Here $d^4x = dx^0 \wedge \ldots \wedge dx^3$, $\quad g = \det g_{\mu\nu} < 0$, so $-g = |g|$. The product $d^4x\sqrt{-g}$ is the <u>invariant</u> <u>volume</u> <u>element</u>, as will be explained later in the course. $\mathcal{L}_G$ must be a scalar in order that the integral be independent of coordinates. The simplest scalar that can be constructed out of $g_{\mu\nu}$ and its derivatives (apart from trivial things like $g^\mu{}_\mu = 4$) is the curvature scalar R. So we guess that $\mathcal{L}_G \propto R$, and we look at the variation,

$$\delta \int d^4x \sqrt{-g}\, R = 0.$$

The variation is carried out by $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$. First we compute the variation in $g^{\mu\nu}$ induced by $\delta g_{\mu\nu}$. Use

$$g^{\mu\alpha} g_{\alpha\beta} = \delta^\mu_\beta \quad \Rightarrow$$

$$\delta g^{\mu\alpha} g_{\alpha\beta} + g^{\mu\alpha} \delta g_{\alpha\beta} = 0$$

$$\Rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

Next we compute $\delta\sqrt{-g}$. Let M be a matrix that depends on a parameter λ. Then we have the useful identity,

$$\frac{d}{d\lambda}(\det M) = (\det M)\,\mathrm{tr}\left(M^{-1}\frac{dM}{d\lambda}\right).$$

Identify M with $g_{\mu\nu}$, $\det M = g$, this implies

$$\delta g = g\left(g^{\mu\nu}\delta g_{\mu\nu}\right),$$

or

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\delta g_{\mu\nu}\right).$$

Finally, we need δR. Start ~~by~~ with $\delta\Gamma^{\mu}_{\alpha\beta}$, the change in the L.C. Γ when $g_{\mu\nu}$ goes to $g_{\mu\nu}+\delta g_{\mu\nu}$. Being the ~~change~~ difference between 2 connections, this is a <u>tensor</u>, which we will write as $(\delta\Gamma)^{\mu}{}_{\alpha\beta}$ to be careful about the positions of the indices. Of course $\Gamma^{\mu}_{\alpha\beta}$ itself is not a tensor.

Now we compute $\delta R^{\mu}{}_{\nu\alpha\beta}$ in terms of $\delta\Gamma$. The expression for R has the structure

$$R = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma,$$

omitting all indices. Therefore

$$\delta R = \partial(\delta\Gamma) - \partial(\delta\Gamma) + (\delta\Gamma)\Gamma + \Gamma(\delta\Gamma) - (\delta\Gamma)\Gamma - \Gamma(\delta\Gamma).$$

We evaluate $\delta R^{\mu}{}_{\nu\alpha\beta}$ at an arbitrary point of the manifold that we call O, $\delta R^{\mu}{}_{\nu\alpha\beta}(O)$. We use Riemann normal coordinates based at O, so $\Gamma^{\mu}_{\nu\alpha}(O) = 0$. Thus

$$\delta R^{\mu}{}_{\nu\alpha\beta}(O) = (\delta\Gamma)^{\mu}{}_{\beta\nu,\alpha}(O) - (\delta\Gamma)^{\mu}{}_{\alpha\nu,\beta}(O),$$

since all 4 terms in Γ-$\delta\Gamma$ vanish. Since $\delta\Gamma$ is a tensor, ~~all~~ both terms above are ordinary derivatives of tensors, evaluated at O.

But in R.N.C., such ord. derivs are equal to covariant derivs, (evaluated at 0). So we can replace the comma with a semicolon. Then we have a relation between two tensors,

$$\delta R^{\mu}{}_{\cdot\nu\alpha\beta}(0) = (\delta\Gamma)^{\mu}{}_{\cdot\beta\nu;\alpha}(0) - (\delta\Gamma)^{\mu}{}_{\cdot\alpha\nu;\beta}(0).$$

But since 0 was arbitrary, this is true at all points,

$$\delta R^{\mu}{}_{\cdot\nu\alpha\beta} = (\delta\Gamma)^{\mu}{}_{\cdot\beta\nu;\alpha} - (\delta\Gamma)^{\mu}{}_{\cdot\alpha\nu;\beta}$$

And since it is a tensor eqn, it is valid in all coordinates (not only RNC).

Now by contracting, we get the variation of the Ricci tensor,

$$\delta R_{\nu\beta} = (\delta\Gamma)^{\alpha}{}_{\cdot\beta\nu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\cdot\alpha\nu;\beta}$$

or juggling indices

$$\delta R_{\mu\nu} = (\delta\Gamma)^{\alpha}{}_{\cdot\nu\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\cdot\alpha\mu;\nu}.$$

Finally, as for the curvature scalar, we have $R = g^{\mu\nu}R_{\mu\nu} = R_{\mu}{}^{\mu}{}_{\cdot}$,

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}$$

$$= -g^{\mu\alpha}g^{\nu\beta}\,\delta g_{\alpha\beta}R_{\mu\nu} + g^{\mu\nu}\left((\delta\Gamma)^{\alpha}{}_{\cdot\nu\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\cdot\alpha\mu;\nu}\right)$$

$$\delta R = -R^{\mu\nu}\,\delta g_{\mu\nu} + (\delta\Gamma)^{\alpha\mu}{}_{\cdot\cdot\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\cdot\alpha}{}^{\mu}{}_{\cdot;\mu}.$$

Thus,

$$\delta\int d^4x\,\sqrt{-g}\,R = \int d^4x\left[\delta\sqrt{-g}\,R + \sqrt{-g}\,\delta R\right]$$

$$= \int d^4x\,\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}R\,\delta g_{\mu\nu} - R^{\mu\nu}\delta g_{\mu\nu} + (\delta\Gamma)^{\alpha\mu}{}_{\cdot\cdot\mu;\alpha} - (\delta\Gamma)^{\alpha}{}_{\cdot\alpha}{}^{\mu}{}_{\cdot;\mu}\right]$$

$$= 4 \text{ terms}.$$

The last two terms vanish on integration. For example, let

$$X^\alpha = \delta\Gamma^{\alpha\mu}{}_{;\mu},$$

so the expression

$$X^\alpha{}_{;\alpha}$$

(the covariant divergence of a vector) appears in the integral. This can also be written,

$$X^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,X^\alpha\right)_{,\alpha}$$

by an identity we will prove shortly. Thus

$$\int d^4x\,\sqrt{-g}\;\; X^\alpha{}_{;\alpha} = \int d^4x\,\left(\sqrt{-g}\,X^\alpha\right)_{,\alpha} = 0$$

by integration by parts (X vanishes at ∞). (or maybe M is compact). Similarly for the 4th term. Thus,

$$\delta\int\sqrt{-g}\,d^4x\;R = \int d^4x\,\sqrt{-g}\left[+\tfrac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right]\delta g_{\mu\nu} = 0$$

for all $\delta g_{\mu\nu}$ $\Rightarrow$

$$+\tfrac{1}{2}g^{\mu\nu}R - R^{\mu\nu} = -G^{\mu\nu} = 0$$

the vacuum Einstein equations.

Conventionally we take

$$\mathcal{L}_G = \frac{R}{16\pi G},$$

G= Newton's constant of gravitation, henceforth set to 1.

If a matter Lagrangian $\mathcal{L}_M$ is added to $\mathcal{L}_G$ and the overall variational principle is

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$$\delta \int d^4x \sqrt{-g}\, (\mathcal{L}_G + \mathcal{L}_M) = 0$$

with $\mathcal{L}_G = R/16\pi$, then to get the right field equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

we must have

$$\frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g}\, \mathcal{L}_M = \frac{1}{2} T^{\mu\nu}.$$

Now we turn to the problem of putting spinors into curved space-time. The idea is to add the Dirac Lagrangian to the gravitational on $\mathcal{L}_G$. In special relativity, (SR), the Dirac Lagrangian is

$$\mathcal{L}_D = \bar{\psi}\,(i\gamma^\alpha \partial_\alpha - m)\,\psi$$

in units $\hbar = c = 1$. Notation is standard, γ^α are the Dirac 4×4 matrices, ψ is the Dirac 4-spinor, and ∂_α means differentiation wrt. flat space coordinates $(t, \vec{x}) = x^\mu$.

There are two problems on putting this into GR. The first is that the usual γ matrices are tied to inertial frames in SR, i.e. coordinates $x^\mu = (t, \vec{x})$. Rather than trying to generalize the γ matrices to other frames, a better choice is to introduce an orthonomal vierbein $\{e_\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu}$, and replace ∂_α by e_α. Then we can use the standard γ matrices of SR even in GR.

The second problem is that $e_\alpha \psi$ ($= \psi_{,\alpha}$ in our generalized comma notation) is not covariant, so $\mathcal{L}_D$ is not a scalar in GR, as written. Obviously we must replace $e_\alpha \psi$

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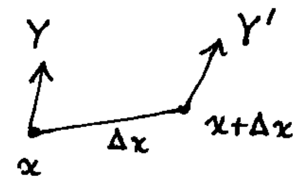
with $\nabla_\alpha \psi$, where $\nabla_\alpha \equiv \nabla_{e_\alpha}$ is a covariant derivative. But how do we compute covariant derivatives of spinors?

Take our clue from the covariant derivative of ordinary vectors. Begin with parallel transport of vector $Y \in T_x M$ to $Y' \in T_{x+\Delta x} M$,



In some local chart x^μ, we know that

$$Y'^\mu = (\delta^\mu_\nu - \Delta x^\sigma \Gamma^\mu_{\sigma\nu}) Y^\nu$$

where $\Gamma^\mu_{\sigma\nu}$ are the connection coefficients w.r.t. the chart x^μ. If we transform this to an ON vierbein $\{e_\alpha\}$, then we have

$$Y'^\alpha = (\delta^\alpha_\beta - \Delta x^\gamma \Gamma^\alpha_{\gamma\beta}) Y^\beta,$$

where now $\Gamma^\alpha_{\gamma\beta}$ is the connec. coeffs. w.r.t. to the vierbein. It is an equation of the same form, in spite of the fact that Γ does not transform as a tensor. Now, hower, the matrix

$$\Lambda^\alpha{}_\beta = (I + \Omega)^\alpha{}_\beta = \delta^\alpha_\beta - \Delta x^\gamma \Gamma^\alpha_{\gamma\beta}$$

is an infinitesimal Lorentz transformation, where the correction term

$$\Omega_{\alpha\beta} = -\Delta x^\gamma \Gamma_{\alpha\gamma\beta}$$

satisfies $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$ (it is an element of the Lie algebra of ~~so~~ $o(3,1)$.)

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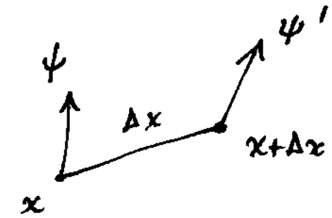
To parallel transport Dirac spinors from x to $x+\Delta x$, say, $\psi \mapsto \psi'$



we may apply the Lorentz transformation $D(\Lambda)$ to ψ, where $\Lambda = I + \Omega$ is the infinitesimal Lorentz transformation defined above. Here $D(\Lambda)$ is the representation of the Lorentz group for Dirac spinors. Actually, it is not a representation, since it is double-valued. More about that next week.