4/13/04

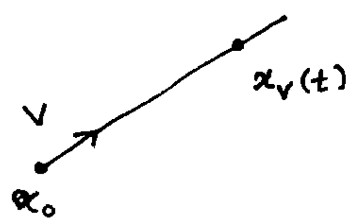
Summary of Riemann normal coordinates.

x_0 = chosen point

$x_V(t)$ = soln of geodesic eqn. $\quad \dfrac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}\dfrac{dx^\alpha}{dt}\dfrac{dx^\beta}{dt} = 0$

subject to init. conds. $\quad x^\mu(0) = x_0^\mu$

$$\frac{dx^\mu}{dt}(0) = V^\mu \quad , \quad V \in T_{x_0}M.$$



For example, if V is a unit vector, $|V|=1$, then $\exp(V)$ is a point on M at unit dist. from x_0.

Facts:

$x_V(t)$ depends only on tV.

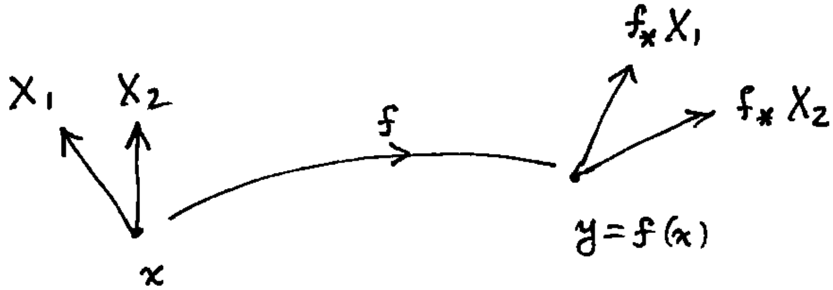
dist $= s = t|V|$.

Define: $\quad \exp: T_{x_0}M \to M : V \longmapsto x_V(1) = \exp(V)$

Let $\{e_\mu\}$ be basis in $T_{x_0}M$, $\quad V = W^\mu e_\mu$. Then assign coords W^μ to point $\exp(V)$ on M. $\quad W^\mu$ are R.N.C. on M.

4/13/04

Now for some remarks about conformal transformations and isometries. Let (M, g) be a (pseudo)-Riemannian manifold, with Levi-Civita connection ∇, ~~can~~ and consider a map $f: M \to M$. [This discussion is easily generalized to the case $f: M \to N$, between two Riem. manifolds.]. Let $y = f(x)$, some $x \in M$, and let $X_1, X_2 \in T_x M$.



We compare the scalar products $g|_x(X_1, X_2)$ and $g|_{f(x)}(f_* X_1, f_* X_2)$. If these are proportional by some positive scale factor, written $e^{2\sigma(x)}$ where $\sigma: M \to \mathbb{R}$ is a scalar field, i.e., if $\exists\, \sigma$ such that

$$g|_{f(x)}(f_* X_1, f_* X_2) = e^{2\sigma(x)}\, g|_x(X_1, X_2)$$

for all $X_1, X_2 \in T_x M$, and all $x \in M$, then we say $f: M \to M$ is a <u>conformal transformation</u>. The condition can be written more compactly as

$$\boxed{f_* g = e^{2\sigma} g.} \qquad \text{(Defn of conformal transf. } f\text{)}.$$

If this equation holds for $\sigma = 0$, i.e., if

$$\boxed{f_* g = g} \qquad \text{(Defn. of isometry } f\text{)}.$$

Then f is said to be an <u>isometry</u>. An isometry is a special

4/13/04

case of a conformal transformation. Under conformal transformation, scalar products are preserved up to a scaling; this preserves angles but not necessarily lengths. Under an isometry, both lengths and angles are preserved.

Historical note: Conformal transformations entered physics with Weyl's 1919 attempt to unify E+M and general relativity (then very new). In Weyl's theory, the integral $\int A_\mu dx^\mu$ of the E+M vector potential was interpreted as a scale factor for a scaling of the metric. The idea failed, however, because this scale factor is path dependent. But this is where the word "gauge" comes from in "gauge transformation." Later $\int A_\mu dx^\mu$ was reinterpreted as part of the phase of the quantum wave function, the change of which is still called a gauge transformation.

In modern times conformal field theories are important as exactly solvable 2D models of quantum field theories, in 2D critical phenomena, and in string theory.

An example of a conformal transformation in 2D is any complex function $w = w(z)$ on the complex plane. Let $z = x+iy$, $w = u+iv$. Then can easily show,

$$dx^2+dy^2 = \frac{du^2+dv^2}{u_x^2+u_y^2}, \qquad u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}$$

because of Cauchy-Riemann conditions. Thus the mapping $z \mapsto w$ is conformal.

Examples of isometries are translations and rotations (the Euclidean group) on Euclidean $\mathbb{R}^n$.

A concept closely related to conformal transformations is the following. Let M be a manifold with two metrics g and $\bar{g}$, and suppose

$$\bar{g} = e^{2\sigma} g.$$

Then g and $\bar{g}$ are said to be conformally related. Let δ be a metric that in some coordinates has the form $g_{\mu\nu} = \delta_{\mu\nu}$ (flat space). Then if $\bar{g} = e^{2\sigma}\delta$, then $\bar{g}$ is said to be conformally flat.

As discussed, the integrability condition for the existence of a coordinate system such that $g_{\mu\nu} = \delta_{\mu\nu}$ is the vanishing of the Riemann tensor, $R^{\mu}{}_{\nu\alpha\beta} = 0$. It turns out that there is another (less restrictive) condition that g satisfies if it is conformally flat, i.e., if there exists a coord. system and scalar field σ such that $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$. This condition is the vanishing of the Weyl tensor,

$$W_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{m-2}\left(g_{\mu\beta}R_{\alpha\nu} - g_{\mu\alpha}R_{\beta\nu} + g_{\alpha\nu}R_{\mu\beta} - g_{\beta\nu}R_{\mu\alpha}\right)$$
$$+ \frac{1}{(m-1)(m-2)}\left(g_{\mu\alpha}g_{\beta\nu} - g_{\mu\beta}g_{\alpha\nu}\right)R,$$

where $m = \dim M$, $R_{\mu\nu}$ = Ricci tensor, R = curvature scalar. W has the property that it is invariant under conformal transformations; hence if g is conformally flat, then $W=0$. Conversely, if $W=0$, then for $m \geq 4$, the metric is conformally flat; but for $m=3$ $W=0$ always, and for $m=2$, any g is conformally flat.

Back to isometries. It's easy to show that given (M, g), the set of isometries forms a group. This can be thought of as an abstract group G whose action Φ_a on M is the set of isometries, that is

$$\Phi_a^* g = g \,, \qquad a \in G.$$

(g = metric, not a group element). Assume that G is a Lie group, and let $a = \exp(tV)$ where $V \in \mathfrak{g}$ = Lie algebra of G. Then

$$\left.\frac{d}{dt}\right|_{t=0} \Phi^*_{\exp(tV)} = V_M = \text{induced vector field, call it } X_V \in \mathfrak{X}(M).$$

This eqn. holds if both sides act on scalars. For other tensors, (such as g) replace RHS by $\mathcal{L}_{X_V}$ (the Lie derivative). Thus if G is the isometry group, then

$$\Phi^*_{\exp(tV)}\, g = g \,,$$

or, applying $\left.\frac{d}{dt}\right|_{t=0}$,

$$\mathcal{L}_{X_V} g = 0.$$

A vector field $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X g = 0$ is called a <u>Killing vector field</u>. Killing vector fields represent infinitesimal isometries. A problem is to find the Killing vector fields given g.

Let X be a Killing vector field. Then by writing $\mathcal{L}_X$ in components, we have

$$(\mathcal{L}_X g)_{\alpha\beta} = X^\mu g_{\alpha\beta,\mu} + X^\mu{}_{,\alpha}\, g_{\mu\beta} + X^\mu{}_{;\beta}\, g_{\alpha\mu} = 0.$$

This is a differential equation that X must satisfy. ~~[illegible]~~ Use the Levi-Civita connection, so that

$$g_{\alpha\beta,\mu} = \Gamma^{\sigma}_{\mu\alpha}\, g_{\sigma\beta} + \Gamma^{\sigma}_{\mu\beta}\, g_{\alpha\sigma}$$

Thus

$$X^{\sigma}_{,\alpha}\, g_{\sigma\beta} + X^{\sigma}_{,\beta}\, g_{\alpha\sigma} + \Gamma^{\sigma}_{\mu\alpha}\, g_{\sigma\beta} X^{\mu} + \Gamma^{\sigma}_{\mu\beta}\, g_{\alpha\sigma}$$

$$= g_{\sigma\beta} X^{\sigma}_{;\alpha} + g_{\alpha\sigma} X^{\sigma}_{;\beta}$$

$$= \boxed{X_{\beta;\alpha} + X_{\alpha;\beta} = 0}$$

This is Killing's equ, nice compact form for equ. that Killing vectors must satisfy.

Here are some examples of Killing vector fields on some spaces. First take Euclidean $\mathbb{R}^n$ in standard coordinates. Then $X_{\beta;\alpha} = X_{\beta,\alpha}$, and Killing's equ. is

$$X_{\beta,\alpha} + X_{\alpha,\beta} = 0 , \quad \text{also } X_\beta = X^\beta \text{ since } g_{\mu\nu} = \delta_{\mu\nu}.$$

Expand X_α in a Taylor series:

$$X_\alpha = a_\alpha + b_{\alpha\beta}\, x^\beta + c_{\alpha\beta\gamma}\, x^\beta x^\gamma + \cdots$$

so

$$\begin{aligned} X_{\alpha,\beta} &= b_{\alpha\beta} + 2\, c_{\alpha\beta\gamma}\, x^\gamma + \cdots \\ X_{\beta\alpha} &= b_{\beta\alpha} + 2\, c_{\beta\alpha\gamma}\, x^\gamma + \cdots \\ \hline 0 &= (b_{\alpha\beta} + b_{\beta\alpha}) + 2\,(c_{\alpha\beta\gamma} + c_{\beta\alpha\gamma})\, x^\gamma + \cdots \end{aligned}$$

Thus $b_{\alpha\beta} = -b_{\beta\alpha}$ (antisymmetric), a_α = any const vector. As for c, it is symmetric in $\beta\gamma$ and antisymm. in $\alpha\beta$, which $\Rightarrow$ $c = 0$. Same for all higher tensors. Thus we find

$$X_\alpha = a_\alpha + b_{\alpha\beta} x^\beta .$$

We ~~may~~ recognize this as an infinitesimal displacement (a_α) composed with an infinitesimal rotation ($b_{\alpha\beta} = -b_{\beta\alpha}$ means $b \in so(n)$). The number of parameters is $m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$ ($m = \dim M$). This is the number of lin. indep. Killing vector fields; it is a finite number. In fact, it is the maximum number that a space of m dimensions can have. For this reason, Euclidean $\mathbb{R}^m$ is called a maximally symmetric space. Another example of a Max-Sym. Space is the sphere S^n, ~~with~~ with the induced metric obtained by embedding in Euclidean $\mathbb{R}^{n+1}$. S^n is invariant under $O(n+1)$, a group with $n(n+1)/2$ dimensions. (Same number). Similarly, n-dim. surfaces of const. neg. curvature can be imbedded in Minkowski $\mathbb{R}^{n+1}$, and are maximally symmetric under the $(n+1)$-dimensional Lorentz group.

Since the Killing vectors are the infinitesimal generators of the ~~group~~ action of the isometry group on M, they must form a Lie algebra. This is easy to show directly. Let X, Y be two Killing vector fields, so

$$\mathcal{L}_X g = 0$$
$$\mathcal{L}_Y g = 0 .$$

Then since $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, we have $\mathcal{L}_{[X,Y]}\, g = 0$, hence $[X,Y]$ is also a Killing vector field.

Can also talk about conformal Killing vectors.

4/13/04

New subject. Now we put the basic equations of metrical geometry into a <u>noncoordinate</u> <u>basis</u>, and also introduce the formalism of Cartan. In 4 dimensions, a noncoordinate basis is sometimes called a <u>tetrad</u> or <u>vierbein</u> (German for "four legs"), because a frame is a set of lin. indep. vectors

a "dreibein"

. In many dimensions, we may refer to the frame as a <u>vielbein</u> (many legs).

Nakahara distinguishes components w.r.t. to a vielbein from those w.r.t. a coordinate basis by using $\alpha, \beta, \gamma \ldots$ for the vielbein and $\mu, \nu, \lambda, \ldots$ for the coordinate basis. We will just use any indices in the following, but it will be understood that we are working in a non-coordinate basis.

Let $\{e_\mu\}$ be the basis vector fields, assumed to be lin. indep. at each point of some region of space. Let $\{\theta^\mu\}$ be the dual basis of 1-forms, so that

$$\theta^\mu(e_\nu) = \delta^\mu_\nu .$$

The basis vectors satisfy

$$[e_\mu, e_\nu] = c^\sigma_{\mu\nu}\, e_\sigma$$

where $c^\sigma_{\mu\nu} = -c^\sigma_{\nu\mu}$ are the structure constants. (not really const. however). Similarly, we have

$$d\theta^\mu = -\tfrac{1}{2}\, c^\mu_{\alpha\beta}\;\theta^\alpha \wedge \theta^\beta .$$

(derived previously). (Here's how:

$$d\theta^\mu(e_\alpha, e_\beta) = e_\alpha[\underbrace{\theta^\mu(e_\beta)}_{\delta^\mu_\beta}] - e_\beta[\underbrace{\theta^\mu(e_\alpha)}_{\delta^\mu_\alpha}] - \theta^\mu([e_\alpha, e_\beta])$$

$$= 0 - 0 - c^\sigma_{\alpha\beta}\,\theta^\mu(e_\sigma) = -c^\mu_{\alpha\beta}.)$$

4/13/04

Also use the notation,

$$e_\alpha f = f_{,\alpha} \qquad \text{for } f \in \mathcal{F}(M).$$

Note: Nakahara avoids this, he always writes things like $e_\alpha[\Gamma^{\mu}_{\sigma\tau}]$ for what I will write as $\Gamma^{\mu}_{\sigma\tau,\alpha}$.

Now begin with (M, ∇), but don't assume any g, nor that torsion $T=0$. ~~Comp~~ First we define connection coefficients,

$$\nabla_\mu \equiv \nabla_{e_\mu}$$

$$\nabla_\mu e_\nu = \Gamma^{\alpha}_{\mu\nu} e_\alpha$$

Equivalently, $\qquad \nabla_\mu \theta^\nu = -\Gamma^{\nu}_{\mu\alpha}\theta^\alpha.$

Now defn of torsion,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. = -T(Y,X) = \text{a vector field.}$$

Components:

$$\begin{aligned} T(e_\mu, e_\nu) &= T^{\alpha}_{\mu\nu} e_\alpha \\ &= \nabla_\mu e_\nu - \nabla_\nu e_\mu - [e_\mu, e_\nu] \\ &= (\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} - C^{\alpha}_{\mu\nu})\, e_\alpha \end{aligned}$$

hence $\qquad \boxed{T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} - C^{\alpha}_{\mu\nu}}$

Similarly for the curvature tensor,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$$R(e_\alpha, e_\beta)e_\nu = R^{\mu}{}_{\nu\alpha\beta}\, e_\mu$$

4/13/04

We found previously, (4/8/04, p.4)

$$R^{\mu}{}_{\cdot\nu\alpha\beta} = \Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta} + \Gamma^{\sigma}_{\beta\nu}\Gamma^{\mu}_{\alpha\sigma} - \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\mu}_{\beta\sigma} - C^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\nu} .$$

Now follow Cartan and make the following definitions:

$$\omega^{\mu}{}_{\cdot\nu} = \Gamma^{\mu}_{\alpha\nu}\,\theta^{\alpha} \qquad \text{(Lie-algebra valued 1-form)}$$

$$T^{\mu} = \tfrac{1}{2}\, T^{\mu}_{\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta} \qquad \text{(vector-valued 2-form)}$$

$$R^{\mu}{}_{\cdot\nu} = \tfrac{1}{2}\, R^{\mu}{}_{\cdot\nu\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta} \qquad \text{(Lie-algebra valued 2-form).}$$

Now take

$$d\theta^{\mu} = -\tfrac{1}{2}\, C^{\mu}_{\alpha\beta}\,\theta^{\alpha}\wedge\theta^{\beta}$$

use components of T, elim. $C^{\mu}_{\alpha\beta}$ in favor of T, Γ,

$$= +\tfrac{1}{2}\left(T^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\beta\alpha}\right)\theta^{\alpha}\wedge\theta^{\beta}$$

equal

$$= T^{\mu} - \omega^{\mu}{}_{\cdot\beta}\wedge\theta^{\beta},$$

or $$\boxed{d\theta^{\mu} + \omega^{\mu}{}_{\cdot\beta}\wedge\theta^{\beta} = T^{\mu}}$$ 1st Cartan structure eqn.

LHS is a kind of covariant derivative of a 1-form.

Next, take defn. $\omega^{\mu}{}_{\cdot\nu} = \Gamma^{\mu}_{\alpha\nu}\,\theta^{\alpha}$, apply d:

$$d\omega^{\mu}{}_{\cdot\nu} = d\left(\Gamma^{\mu}_{\alpha\nu}\,\theta^{\alpha}\right) = \Gamma^{\mu}_{\alpha\nu,\beta}\,\theta^{\beta}\wedge\theta^{\alpha} + \Gamma^{\mu}_{\alpha\nu}\,d\theta^{\alpha}$$

$$= \tfrac{1}{2}\left(\Gamma^{\mu}_{\alpha\nu,\beta} - \Gamma^{\mu}_{\beta\nu,\alpha}\right)\theta^{\beta}\wedge\theta^{\alpha} - \tfrac{1}{2}\,\Gamma^{\mu}_{\alpha\nu}\, C^{\alpha}_{\sigma\tau}\,\theta^{\sigma}\wedge\theta^{\tau}$$