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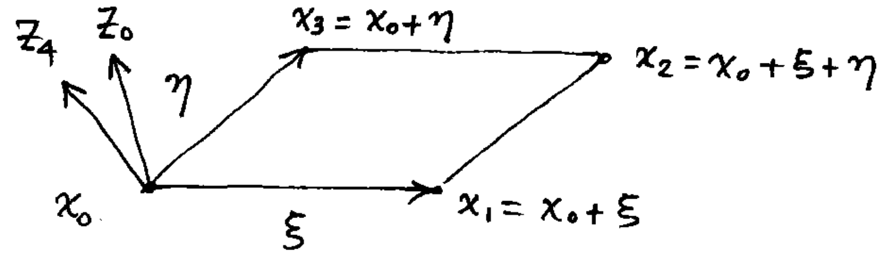
Computation of curvature tensor. Given manifold M with connection ∇, but not nec. anything else (such as g). Parallel transport a vector Z around the 4 sides of a parallelogram spanned by two infinitesimal vectors ξ, η, with corners x_0, x_1, x_2, x_3, giving $Z_0 \to Z_1 \to Z_2 \to Z_3 \to Z_4$, where $Z_0, Z_4 \in T_{x_0}M$:



The transport $Z_0 \to Z_4$ must be linear and near-identity (since the parallelogram is small). Write it

$$Z_4^{\mu} = [\mathrm{Id} - R(\xi,\eta)]^{\mu}{}_{\nu} Z_0^{\nu}$$

where $R(\xi,\eta)^{\mu}{}_{\nu}$ is linear in ξ, η, so

$$R(\xi,\eta)^{\mu}{}_{\nu} = R^{\mu}{}_{\nu\alpha\beta}\, \xi^{\alpha}\eta^{\beta}$$

defines the components $R^{\mu}{}_{\nu\alpha\beta}$ of the curvature tensor. It is anti-symmetric in ξ, η, since if $\xi = \eta$, then you are parallel transporting along a line and back again, which cancels (gives $R(\xi,\xi) = 0$). Thus we expect

$$R^{\mu}{}_{\nu\alpha\beta} = - R^{\mu}{}_{\nu\beta\alpha}.$$

Equivalently, R is a <u>Lie-algebra valued 2-form</u>. Of course we must verify these expected properties of R (such as the fact that it is a tensor).

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On the leg $x_0 \to x_1$, the $\parallel$-transport equation can be solved in a Taylor series in the small displacement ξ, which we expand through 2nd order:

not covariant deriv., just shorthand.

$$Z_1 = \left[\mathrm{Id} - \Gamma_\xi(x_0) + \tfrac{1}{2}\left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2 \right) \right] Z_0$$

shorthand for $Z_1^\mu = [\ \ ]^\mu{}_\nu\, Z_0^\nu$

where $(\Gamma_\xi)^\mu{}_\nu = \xi^\alpha\, \Gamma^\mu_{\alpha\nu}$

$$(\xi\cdot\nabla\Gamma_\xi)^\mu{}_\nu = \xi^\beta \xi^\alpha\, \Gamma^\mu_{\alpha\nu,\beta}.$$

Similarly,

$$Z_3 = \left[\mathrm{Id} + \Gamma_\xi (x_0 + \xi + \eta) + \tfrac{1}{2}\left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)\right] Z_2$$

$$Z_4 = \left[\mathrm{Id} + \Gamma_\eta (x_0 + \eta) + \tfrac{1}{2}\left(-\eta\cdot\nabla\Gamma_\eta + \Gamma_\eta^2\right)\right] Z_3.$$

Now multiply matrices,

$$Z_4 = \underbrace{[\ ][\ ][\ ][\ ]}\, Z_0$$

$$= \mathrm{Id} - \left[\Gamma_\xi\Gamma_\eta - \Gamma_\eta\Gamma_\xi + \xi\cdot\nabla\Gamma_\eta - \eta\cdot\nabla\Gamma_\xi \right]$$

$$= \mathrm{Id} - R(\xi, \eta).$$

From this can read off components of R,

$$\boxed{R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\nu} + \Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\alpha\nu,\beta}}$$

Expression of curvature tensor in terms of connection, in a coordinate basis $e_\mu = \partial/\partial x^\mu$. (No metric required.)

Γ is not a tensor, but R should be a tensor, based on its definition (its a mapping of $T_x M$ onto itself, given ξ and η). A direct proof that R transforms as a tensor is tedious, however.

A coordinate-free approach is better. We define

$$R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

notation $R(X,Y,Z) = \underbrace{R(X,Y)}Z$

will turn out to be same $R(X,Y)$ defined above

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where

$$R(X,Y): \mathfrak{X}(M) \to \mathfrak{X}(M),$$

$$\boxed{R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}}$$ coordinate-free defn of R.

First show that this is a tensor. Must be pointwise linear in X, Y. Because of antisymmetry, suffices to check only X. Let $f \in \mathfrak{F}(M)$

$$R(fX, Y) = \nabla_{fX}\nabla_Y - \nabla_Y \nabla_{fX} - \nabla_{[fX,Y]}$$

$$\to f \nabla_X \nabla_Y$$

$$\to -\nabla_Y f \nabla_X = \cancel{-(Yf)\nabla_X} - f\nabla_Y\nabla_X$$

$$\to = -\nabla_{f[X,Y] - (Yf)X} = -f\nabla_{[X,Y]} + \cancel{(Yf)\nabla_X}$$

$$\to = f\,R(X,Y).$$

Should also be linear in Z. Check it.

$$R(X,Y)fZ = \nabla_X\nabla_Y fZ - \nabla_Y\nabla_X fZ - \nabla_{[X,Y]} fZ$$

$$= T_1 + T_2 + T_3 \quad \text{(3 terms)}.$$

$$T_1 = \nabla_X (Yf) Z + \nabla_X f \nabla_Y Z$$

$$= (XYf) Z + (Yf) \nabla_X Z + (Xf) \nabla_Y Z + f \nabla_X \nabla_Y Z$$

$T_2 = T_1$ with $X \leftrightarrow Y$, minus sign.

$$= -(YXf) Z - (Xf) \nabla_Y Z - (Yf) \nabla_X Z - f \nabla_Y \nabla_X Z$$

$$T_3 = -([X, Y] f) Z - f \nabla_{[X,Y]} Z$$

add em up, get $f R(X,Y) Z$.

So R is a tensor.

Now compute the components of R (starting from coordinate-free definition) and compare to earlier coordinate-based calculation. For variety do this in a non-coordinate basis e_μ $\left(\neq \frac{\partial}{\partial x^\mu}\right)$.

Define:

① $f_{,\mu} = (e_\mu f)$ when $f \in \mathcal{F}(M)$, generalizes notation $f_{,\mu} = e_\mu f = \frac{\partial f}{\partial x^\mu}$ for a coordinate basis.

② $\nabla_\mu = \nabla_{e_\mu}$

③ $\nabla_\alpha e_\beta = \Gamma^\mu_{\alpha\beta} e_\mu$

④ $R(e_\alpha, e_\beta) e_\nu = R^\mu{}_{\nu\alpha\beta} e_\mu$

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$$R(e_\alpha, e_\beta)\, e_\nu = \nabla_\alpha \nabla_\beta e_\nu - \nabla_\beta \nabla_\alpha e_\nu - \nabla_{[e_\alpha, e_\beta]} e_\nu$$

$$\nabla_\alpha \nabla_\beta e_\nu = \nabla_\alpha (\Gamma^\sigma_{\beta\nu}\, e_\sigma) = \Gamma^\sigma_{\beta\nu,\alpha}\, e_\sigma + \Gamma^\sigma_{\beta\nu}\, \Gamma^\mu_{\alpha\sigma}\, e_\mu$$

$- \nabla_\beta \nabla_\alpha e_\nu =$ − same w. ($\alpha \Leftrightarrow \beta$).

$$- \nabla_{[e_\alpha, e_\beta]} e_\nu = - \nabla_{c^\sigma_{\alpha\beta} e_\sigma} e_\nu = - c^\sigma_{\alpha\beta}\, \nabla_\sigma e_\nu = - c^\sigma_{\alpha\beta}\, \Gamma^\mu_{\sigma\nu}\, e_\mu$$

$c^\sigma_{\alpha\beta}$ = structure consts of basis.

gives

$$\boxed{R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu_{\beta\nu,\alpha} + \Gamma^\sigma_{\beta\nu}\,\Gamma^\mu_{\alpha\sigma} - \Gamma^\mu_{\alpha\nu,\beta} - \Gamma^\sigma_{\alpha\nu}\,\Gamma^\mu_{\beta\sigma} - c^\sigma_{\alpha\beta}\,\Gamma^\mu_{\sigma\nu}}$$

agrees with earlier calculation in coord. basis, for which $c^\sigma_{\alpha\beta} = 0$.

In the case of the Levi-Civita connection on a (pseudo)-Riemannian manifold, the curvature tensor is also called the Riemann tensor. I'm not sure if that is appropriate in other cases.

The curvature tensor has various symmetries, depending on the assumptions. In the most general case (manifold M + connection ∇, nothing else) we have the symmetry,

$$R(X,Y) = -R(Y,X) \quad \text{or} \quad R^\mu{}_{\nu\alpha\beta} = -R^\mu{}_{\nu\beta\alpha}.$$

This just says that R is a 2-form (indices α, β). That's all in this case.

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If in addition we assume torsion $T=0$, then there are two further symmetries, called the 1st and 2nd Bianchi identities. (These do not require the existence of a metric.)

The first Bianchi identity is an algebraic condition. Use $T=0$.

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0.$$

apply ∇_Z:

$$\nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \nabla_Z [X,Y] = 0.$$

use

$$T(Z,[X,Y]) = \nabla_Z [X,Y] - \nabla_{[X,Y]} Z - [Z,[X,Y]] = 0.$$

Jacobi ident.

so

$$\nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \nabla_{[X,Y]} Z - [Z,[X,Y]] = 0.$$

cycle X,Y,Z

$$\nabla_X \nabla_Y Z - \nabla_X \nabla_Z Y - \nabla_{[Y,Z]} X - [X,[Y,Z]] = 0$$

$$\nabla_Y \nabla_Z X - \nabla_Y \nabla_X Z - \nabla_{[Z,X]} Y - [Y,[Z,X]] = 0.$$

add:

$$\boxed{\begin{aligned} &R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \\ \text{or}\quad &R^\mu{}_{\nu\alpha\beta} + R^\mu{}_{\alpha\beta\nu} + R^\mu{}_{\beta\nu\alpha} = 0 \end{aligned}}$$

1st Bianchi, valid when $T=0$.

write this last equation as $R^\mu{}_{[\nu\alpha\beta]} = 0$ $\quad$ [ ] means antisymmetrize.

Now 2nd Bianchi identity, which is a differential equation satisfied by R. Again begin with $T=0$:

$$T(X,Y) = 0$$

$$\Rightarrow \quad R(T(X,Y),Z) = 0$$

$\Rightarrow$ (next page)

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$$R(\nabla_X Y, Z) - R(\nabla_Y X, Z) - R([X,Y],Z) = 0.$$

cycle, add $$R(\nabla_Y Z, X) - R(\nabla_Z Y, X) - R([Y,Z],X) = 0$$

$$R(\nabla_Z X, Y) - R(\nabla_X Z, Y) - R([Z,X],Y) = 0$$

Result equiv. to

$$\underbrace{R(\nabla_Z X, Y) + R(X, \nabla_Z Y)} - R([X,Y],Z) + \circlearrowleft = 0.$$

now,

$$\nabla_Z R(X,Y) = (\nabla_Z R)(X,Y) + \overbrace{R(\nabla_Z X, Y) + R(X, \nabla_Z Y)}^{\text{same}} + R(X,Y)\nabla_Z$$

so $$\Rightarrow [\nabla_Z, R(X,Y)] - (\nabla_Z R)(X,Y) - R([X,Y],Z) + \circlearrowleft = 0.$$

Now 1st + 3rd terms (+ $\circlearrowleft$) cancel:

$$[\nabla_Z, R(X,Y)] - R([X,Y],Z) + \circlearrowleft$$

by Jacobi

$$= \nabla_Z\nabla_X\nabla_Y - \nabla_Z\nabla_Y\nabla_X - \cancel{\nabla_Z\nabla_{[X,Y]}} - \cancel{\nabla_{[X,Y]}}\nabla_Z + \cancel{\nabla_Z\nabla_{[X,Y]}} + \cancel{\nabla_{[[X,Y],Z]}}$$

$$- \nabla_X\nabla_Y\nabla_Z + \nabla_Y\nabla_X\nabla_Z + \cancel{\nabla_{[X,Y]}}\nabla_Z + \circlearrowleft$$

$$= \nabla_Z\nabla_X\nabla_Y - \nabla_Z\nabla_Y\nabla_X - \nabla_X\nabla_Y\nabla_Z + \nabla_Y\nabla_X\nabla_Z$$

$$+ \nabla_X\nabla_Y\nabla_Z - \nabla_X\nabla_Z\nabla_Y - \nabla_Y\nabla_Z\nabla_X + \nabla_Z\nabla_Y\nabla_X$$

$$+ \nabla_Y\nabla_Z\nabla_X - \nabla_Y\nabla_X\nabla_Z - \nabla_Z\nabla_X\nabla_Y + \nabla_X\nabla_Z\nabla_Y = 0.$$

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So,

$$\boxed{(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(X,Z) = 0}$$

Digression on rules for semicolon. Consider the covariant derivative of a vector field.

$$(\nabla_X Y)^\mu = X^\alpha (Y^\mu{}_{,\alpha} + \Gamma^\mu_{\alpha\beta} Y^\beta).$$

Write this as $X^\alpha Y^\mu{}_{;\alpha}$, thereby defining $Y^\mu{}_{;\alpha} = Y^\mu{}_{,\alpha} + \Gamma^\mu_{\alpha\beta} Y^\beta$. Similarly for other tensors.

Then 2nd Bianchi can be written,

$$R^\mu{}_{\nu[\alpha\beta;\gamma]} = 0$$

Looks sort of like the vanishing exterior derivative of a 2-form (that's what it would be if R did not have indices μ,ν), but we haven't yet defined what we mean by the exterior derivatives of Lie algebra valued forms (more on that later). In any case, notice that it only holds when $T=0$.

If M also has a metric g, then there will be further symmetries if $\nabla g = 0$ (a metric connection). So now assume we have (M, ∇, g) with $\nabla g = 0$ (but not nec. $T=0$). Since we have a g, we can raise + lower indices, get things like $R_{\mu\nu\alpha\beta}$ (purely covariant). Then $\nabla g = 0$ means that the infinitesimal transformations generated on going around a small parallelogram are actually <u>orthogonal transformations</u>, hence the Lie algebra consists of antisymmetric matrices, hence

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} \qquad \text{(metric connection)}.$$

Finally, if we assume both a metric connection $\nabla g = 0$ and vanishing torsion $T = 0$, i.e., the Levi-Civita connection, then there is another symmetry,

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}.$$

You can compute the number of independent components of $R^{\mu}{}_{\nu\alpha\beta}$ under various assumption. For example, if you just have a connection and nothing else there are

$$\frac{n^3(n-1)}{2} \qquad n = \dim M.$$

indep. components (only symmetry is 2-form symmetry). But with the Levi-Civita connection (do the combinatorics) you find

$$\frac{n^2(n^2-1)}{12}.$$

indep. components. Table:

n	$\frac{n^2(n^2-1)}{12}$
0	0
1	0
2	1
3	6
4	20

Easy to understand ~~of~~ why only one component for $n=2$. A 2-form on a 2D space has only one indep. components, and $SO(2)$ is 1-dimensional. In this case, the holonomy around any loop (infinitesimal or otherwise) is specified by an angle of rotation θ, and R can be reduced to a scalar-valued 2-form (the usual kind). Thus, it represents a kind of "angle density" on M, the integral of this over some area gives the holonomy on parallel transporting around the boundary. For example, on the usual 2-sphere with the obvious metric and connection, you find

$$R = \sin\theta \, d\theta \wedge d\phi = \text{"} d\Omega \text{"}$$

the solid angle.

In higher dimensions the holonomy group is usually non Abelian, so you can't get the ~~angle of~~ holonomy on going around a finite loop by integrating the 2-form (R) over the interior.

Some tensors important in GR. → assume LC connection. By contracting the Riemann tensor, we get the Ricci tensor, which can be contracted to the curvature scalar:

Ricci: $$R_{\nu\beta} = R^{\mu}{}_{\nu\mu\beta}$$

Curv. Scalar: $$R = R^{\nu}{}_{\cdot\nu} = R^{\mu\nu}{}_{\cdot\cdot\mu\nu}.$$

The 2nd Bianchi identity implies a differential equation satisfied by these:

$$R^{\mu}{}_{\nu\alpha\beta;\gamma} + R^{\mu}{}_{\cdot\nu\gamma\alpha;\beta} + R^{\mu}{}_{\cdot\nu\beta\gamma;\alpha} = 0 \qquad (\text{contract } \mu, \alpha)$$

$$R_{\nu\beta;\gamma} + \underbrace{R^{\mu}{}_{\nu\gamma\mu;\beta}}_{= -R^{\mu}{}_{\nu\mu\gamma;\beta} = -R_{\nu\gamma;\beta}} + R^{\mu}{}_{\cdot\nu\beta\gamma;\mu} = 0$$

$$R_{\nu\beta;\gamma} - R_{\nu\gamma;\beta} + R^{\mu}{}_{\cdot\nu\beta\gamma;\mu} = 0. \qquad (\text{contract } \nu, \beta)$$

$$R_{;\gamma} - R^{\nu}{}_{\gamma;\nu} + \underbrace{R^{\mu\nu}{}_{\nu\gamma;\mu}}_{= -R^{\nu\mu}{}_{\nu\gamma;\mu} = -R^{\mu}{}_{\gamma;\mu}} = 0$$

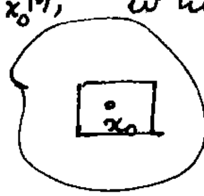
$$\Rightarrow \quad R^{\mu}{}_{\nu;\mu} - \frac{1}{2} R_{;\mu} = 0.$$

Shows appearance of Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$

satisfies $G^{\mu}{}_{\nu;\mu} = 0$, necessary for field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ since $T^{\mu}{}_{\nu;\mu} = 0$ (local energy-momentum conservation).

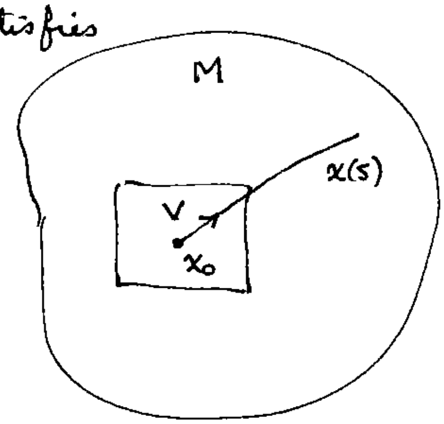
Now we consider <u>Riemann normal coordinates</u>. These are coordinates that simplify the expressions for tensors and covariant derivatives as much as possible in a neighborhood of a given point. We know that a small piece of M is approximately flat. Riemann normal coordinates take advantage of this to make various expression look as much as possible like those on a flat space. The idea is the following. The tangent space $T_{x_0}M$ looks like a small piece of M in the neighborhood of x_0. We can impose linear coordinates on $T_{x_0}M$, which is a vector space. Can those coordinates somehow be extended to make coordinates on M itself?



Let $V \in T_{x_0}M$ be a vector in the "initial" tangent space (at x_0), and consider the geodesic $x(s)$ that satisfies

$$x(0) = x_0$$

$$\frac{dx}{ds}(0) = V.$$



Let the point reached after elapsed parameter s be $p(V, s)$.

The equation of the geodesic is

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$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}(x) \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0.$$

Here x^μ = any coordinates in a neighborhood of x_0.

This equation is homogeneous in s, so if $x^\mu(s)$ is a solution, then so is $x^\mu(ks)$ for $k \in \mathbb{R}$. But they satisfy different initial conditions,

If $\left.\frac{dx^\mu(s)}{ds}\right|_{s=0} = V^\mu$, then $\left.\frac{d}{ds}\right|_{s=0} x^\mu(ks) = kV^\mu$.

In other words, if you scale the initial vector by k, then the curve is traversed k times as fast. In other words,

$$p(kV, s/k) = p(V, s).$$

Thus $p(V,s) = p(sV, 1)$. The point $p(V,s)$ actually depends only on the product sV. Thus we can define,

$$\exp : T_{x_0}M \to M : V \mapsto p(V,1).$$

This map is onto in some neighborhood of x_0, which is "obvious" if you think of $T_{x_0}M$ being a small piece of M (for small vectors in $T_{x_0}M$).

Now choose a basis in $T_{x_0}M$ (could be $\partial/\partial x^\mu|_{x_0}$), with respect to which V has coordinates V^μ. Then use exp to map these coordinates on $T_{x_0}M$ onto M itself. But change the symbol to w^μ, to avoid confusion with coords V^μ on $T_{x_0}M$. Then the geodesics are just <u>radial lines</u> in the w^μ-coordinates,

$$w^\mu(t) = t\,\xi^\mu \qquad \text{(eqn. of geodesic in } w^\mu \text{ coords).}$$

The coordinates w^μ are called <u>Riemann normal coordinates</u>.

Various tensor fields simplify in RNC. Consider first Γ. The eqn. of a geodesic in RNC is

$$\frac{d^2 W^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}(t\xi)\frac{dW^\alpha}{dt}\frac{dW^\beta}{dt} = 0$$

$$\text{or} \quad \Gamma^\mu_{\alpha\beta}(t\xi)\,\xi^\alpha\xi^\beta = 0.$$

setting $t=0$ gives $\Gamma^\mu_{\alpha\beta}(0)\,\xi^\alpha\xi^\beta = 0.$

This holds for all ξ, and since $\Gamma^\mu_{\alpha\beta}$ is symmetric in $(\alpha\beta)$ (Levi-Civita connection), it follows that

$$\Gamma^\mu_{\alpha\beta}(0) = 0 \qquad \text{in RNC.}$$

From this it follows that

$$g_{\mu\nu,\alpha}(0) = 0 \qquad \text{since } \nabla g = 0.$$

So if you expand the metric tensor in RNC about x_0, you find only 2nd order corrections. In general the components $g_{\mu\nu}$ cannot be constant (that would imply a flat space), but in the right coordinates (namely, RNC) they can be made constant through first order terms in the displacement. The Riemann tensor does not vanish at x_0 (it cannot, in general, since it is a tensor), but the expression in terms of the metric simplifies since $\Gamma^\mu_{\alpha\beta}(0)=0$. You find

$$R^\mu{}_{\nu\alpha\beta}(0) = \Gamma^\mu_{\beta\nu,\alpha}(0) - \Gamma^\mu_{\alpha\nu,\beta}(0).$$

The vanishing of the Riemann tensor is the integrability condition that there should exist a coordinate system in which $g_{\mu\nu} = \text{const}$. This would be a flat space.