0
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$$\left.\begin{array}{ll} & \nabla : \quad T_x M \times \mathfrak{X}(M) \to T_x M \\ \text{or} & \nabla : \quad \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \end{array}\right\} : (X, Y) \mapsto \nabla_X Y$$

1. $\nabla_{fX} Y = f \nabla_X Y$

2. $\nabla_{(X_1+X_2)} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

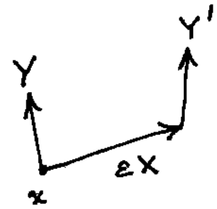
3. $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

4. $\nabla_X (f Y) = (Xf)\, Y + f \nabla_X Y.$

$\nabla_\mu \equiv \nabla_{e_\mu}$

$\nabla_\mu e_\nu = e_\alpha \Gamma^\alpha_{\mu\nu}$

(defn. of Γ).



$Y'^\mu = Y^\mu - \Gamma^\mu_{\alpha\beta} \varepsilon X^\alpha Y^\beta$

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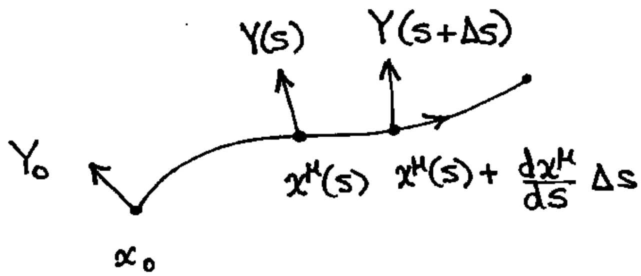


We let $Y(s)$ be the parallel transported vector, $Y(s) \in T_{x(s)}M$.

Then

$$Y^\mu(s+\Delta s) = Y^\mu(s) + \frac{dY^\mu}{ds}\Delta s$$

$$= Y^\mu(s) - \Gamma^\mu_{\alpha\beta}\left(\frac{dx^\alpha}{ds}\Delta s\right)Y^\beta .$$

$$\Rightarrow \quad \boxed{\frac{dY^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}Y^\beta = 0}$$

Eqn. of parallel transport.

homog. in s, transport indep. of param. s of curve.

A differential eqn. that can be solved subject to init conds. $Y(0) = Y_0$.

Can also write it,

$$\boxed{\nabla_{\frac{d}{ds}} Y = 0}$$

where $\frac{d}{ds}$ is the tangent vector along the curve.

An interesting vector to parallel transport is the tangent vector itself. If the parallel transport of the tangent vector is the same as the tangent vector itself, then this is a special property of the curve. Then we have

$$\boxed{\begin{gathered} \nabla_{\frac{d}{ds}}\frac{d}{ds} = 0 \\ \text{or} \quad \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \end{gathered}}$$

Eqn. of a <u>geodesic</u>.

2nd order ode, requires $x^\mu(0)$, $\frac{dx^\mu}{ds}(0)$.

More exactly, this is what may be called a "connection geodesic." If the space is a Riemannian manifold, you can also have a metrical geodesic, the shortest curve between two points. These two need not be the same (indeed M can have a connection without having a metric). But for special connections on Riemannian manifolds, the two kinds of geodesics are identical.

Now we extend ∇ to other types of tensors (besides vectors). We postulate:

$$\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2) \qquad \text{(Leibnitz)}$$

and $\quad \nabla_X \delta = 0 \qquad$ (Kronecker δ).

For example, with $T_1 = f$ (a scalar) and $T_2 = Y$ (a vector field), we have

$$\nabla_X (fY) = (\nabla_X f) Y + f \nabla_X Y,$$

which, comparing to results above, shows that

$$\boxed{\nabla_X f = Xf}$$

(for scalars, the covariant derivative is the obvious convective derivative).

Next, we can work out the action of ∇_X on a 1-form ω by using the rules above:

$$\nabla_X [\omega(Y)] = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$$

$$\text{LHS} = X[\omega(Y)] = X^\mu (\omega_\nu Y^\nu)_{,\mu} = X^\mu (\omega_{\nu,\mu} Y^\nu + \omega_\nu Y^\nu_{,\mu})$$

cancel

$$\text{RHS} = (\nabla_X \omega)_\nu Y^\nu + \omega_\nu X^\mu (Y^\nu_{,\mu} + \Gamma^\nu_{\mu\alpha} Y^\alpha).$$

So we can solve for $(\nabla_X \omega)_\nu$, get

$$\boxed{(\nabla_X \omega)_\nu = X^\mu (\omega_{\nu,\mu} - \Gamma^\alpha_{\mu\nu}\, \omega_\alpha)}$$

c.f. earlier result for vectors

$$\boxed{(\nabla_X Y)^\nu = X^\mu (Y^\nu_{,\mu} + \Gamma^\nu_{\mu\alpha}\, Y^\alpha)}$$

Similarly can work out rules for covariant derivatives (in components) for an arbitrary tensor. Basically you get an ordinary derivative with one correction term with Γ and a + sign for every contravariant index, and one correc. term with Γ and a − sign for every covariant index. For example, you find for the metric tensor,

$$(\nabla_X g)_{\mu\nu} = X^\alpha (g_{\mu\nu,\alpha} - \Gamma^\beta_{\alpha\mu}\, g_{\beta\nu} - \Gamma^\beta_{\alpha\nu}\, g_{\mu\beta})$$

Note, also have

$$\boxed{\nabla_\mu\, dx^\nu = -\Gamma^\nu_{\mu\alpha}\, dx^\alpha}$$

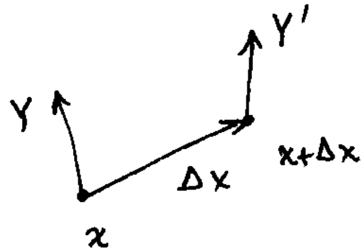
Now we turn to the <u>transformation properties</u> of the connection coefficients $\Gamma^\mu_{\alpha\beta}$. Basic fact is that $\Gamma^\mu_{\alpha\beta}$ is <u>not a tensor</u>. A tensor is a mapping of vectors and covectors onto scalars, that is point-wise linear (linear at each point). We can think of Γ as such a mapping,

$$\Gamma: \mathfrak{X}^*(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M) : (\alpha, X, Y) \mapsto \alpha(\nabla_X Y),$$

$$\Gamma^\mu_{\alpha\beta} = dx^\mu(\nabla_\alpha e_\beta) \qquad e_\beta = \frac{\partial}{\partial x^\beta}.$$

But it is not point-wise linear in the Y operand (it depends on the derivatives of Y as well as the value of Y at a point). Here are various ways to see this.

(1) Consider the parallel transport of Y from x to $x+\Delta x$,



We have

$$Y'^{\mu} = \underbrace{\left(\delta^{\mu}_{\nu} + \Delta x^{\alpha}\, \Gamma^{\mu}_{\alpha\nu} \right)}_{} Y^{\nu}$$

→ a near-identity element of $GL(n,\mathbb{R})$, so we can think of

$$\Gamma^{\mu}_{\cdot\,\nu} = dx^{\alpha}\, \Gamma^{\mu}_{\cdot\,\nu} \quad \text{as a } \underline{\mathfrak{gl}(n,\mathbb{R})\text{-valued 1-form}}.$$

But notice that the <u>components</u> of this 1-form ~~[illegible]~~ depend on the basis chosen in <u>two different</u> tangent spaces (at x and $x+\Delta x$). You can change one without changing the other. Hence $\Gamma^{\mu}_{\alpha\nu}$ does not transform as a tensor.

To emphasize this, consider the following fact: The <u>difference</u> between two connections, say, $\Gamma - \bar{\Gamma}$, <u>is a tensor</u>. That's because $\Gamma - \bar{\Gamma}$ can be thought of as specifying the parallel transport from x to $x+\Delta x$, using Γ, then back again, using $\bar{\Gamma}$. The vector is transported from one tangent space back to the <u>same</u> tangent space. (Say, $Y \to Y' \to Y''$). Then

$$Y''^{\mu} = \left(\delta^{\mu}_{\nu} + dx^{\alpha}\, \Gamma^{\mu}_{\alpha\nu} - dx^{\alpha}\, \bar{\Gamma}^{\mu}_{\alpha\nu} \right) Y^{\nu}.$$

Thus, only one basis (in T_xM) need be chosen to specify the near-identity element of $GL(n,\mathbb{R})$ mapping Y to Y''.

② Just do a brute-force transformation of the connection coefficients. Let

$$e_\alpha = \frac{\partial}{\partial x^\alpha} \qquad e'_\mu = \frac{\partial}{\partial x'^\mu}, \qquad \nabla_\alpha = \nabla_{e_\alpha}$$
$$\nabla'_\mu = \nabla_{e'_\mu}$$

$$\Gamma^\alpha_{\beta\gamma} = (dx^\alpha, \nabla_\beta e_\gamma)$$

$$= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\gamma} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\sigma}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\beta}\Gamma'^\mu_{\nu\sigma}$$

The 2nd term looks like a tensor transformation law, but the first term spoils it (and involves 2nd derivatives of the coordinate transformation). But if you subtract the transformation laws for two Γ's, say, $\Gamma - \bar{\Gamma}$, then the first term cancels.

Transformation laws like this are familiar for the gauge potential A^a_μ of gauge-field theories (Yang-Mills, QCD).

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besides subtracting $\Gamma - \bar{\Gamma}$

Notice that another way to cancel the first term is to antisymmetrize in (β,γ). This leads to a tensor called the torsion:

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}.$$

This is the component definition of the torsion. The coordinate-free definition is

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) : \quad (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y].$$

This is obviously antisymmetric. To show that it is a tensors we must show that it is linear in both operands (but due to the antisymmetry, we need only check one). Let $f \in \mathcal{F}(M)$. Then

$$
\begin{aligned}
T(fX, Y) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\
&= f \nabla_X Y - \cancel{(Yf)X} - f \nabla_Y X - fXY + \underbrace{YfX}_{\cancel{(Yf)X} + fYX} \\
&= f\, T(X, Y).
\end{aligned}
$$

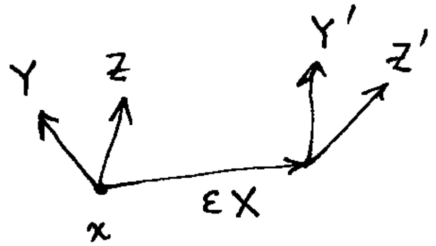
So it's a tensor. Now let $e_\mu = \frac{\partial}{\partial x^\mu}$ be a coordinate basis. Then we define the components of T by

$$
\begin{aligned}
T(e_\alpha, e_\beta) &= T^{\mu}{}_{\alpha\beta}\, e_\mu \\
&= \nabla_\alpha e_\beta - \nabla_\beta e_\alpha - \cancelto{0}{[e_\alpha, e_\beta]} \\
&= (\Gamma^{\mu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\beta\alpha})\, e_\mu,
\end{aligned}
$$

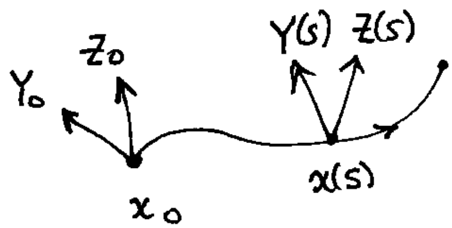
agrees with earlier component definition of T.

We emphasize that a <u>metric</u> and a <u>connection</u> are two different geometrical constructions. You can have a manifold with a connection but without a metric. However, if you <u>do</u> have both a metric and a connection, then you can compute the covariant derivative of the metric, $\nabla_X g$ (along some X).

A connection for which $\nabla_X g = 0$ (for all X) is called a <u>metric connection</u>. If you have a metric connection, then the scalar product of parallel transported vectors is constant: Let $Y, Z \in T_x M$ be two vectors parallel transported along εX to a new point $x + \varepsilon X$, to give new vectors Y', Z':



Then $g(Y,Z) = g(Y',Z')$, if you use a metric connection. Similarly, if $Y(s)$, $Z(s)$ are the parallel transports of $Y_0, Z_0 \in T_{x_0} M$ along a curve,



then

$$\frac{d}{ds}\left[Y(s)_\mu \, Z(s)^\mu\right] = 0.$$

The condition $\nabla_X g = 0$, $\forall X$, implies:

$$g_{\mu\nu,\alpha} = \Gamma^\beta_{\alpha\mu}\, g_{\beta\nu} + \Gamma^\beta_{\alpha\nu}\, g_{\mu\beta}.$$

This equ. can be solved ~~in terms~~ for Γ in terms of g and its derivatives and the torsion tensor. First define

$$\Gamma_{\mu\alpha\beta} = g_{\mu\nu}\, \Gamma^\nu_{\alpha\beta}.$$

Then write

$$S_{\mu\alpha\beta} = \Gamma_{\mu\alpha\beta} + \Gamma_{\mu\beta\alpha}$$

$$T_{\mu\alpha\beta} = \Gamma_{\mu\alpha\beta} - \Gamma_{\mu\beta\alpha}$$

these are the symmetric and antisymmetric parts of Γ. The antisymmetric part is the same as the torsion tensor (but the symmetric part is not a tensor). So we have

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(S_{\mu\alpha\beta} + T_{\mu\alpha\beta}\right)$$

$$g_{\mu\nu,\alpha} = \Gamma_{\nu\alpha\mu} + \Gamma_{\mu\alpha\nu}$$

$$= \frac{1}{2}\left(S_{\nu\alpha\mu} + S_{\mu\alpha\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right)$$

$$g_{\nu\alpha,\mu} = \frac{1}{2}\left(S_{\alpha\mu\nu} + S_{\nu\mu\alpha} + T_{\alpha\mu\nu} + T_{\nu\mu\alpha}\right)$$

$$g_{\alpha\mu,\nu} = \frac{1}{2}\left(S_{\mu\nu\alpha} + S_{\alpha\nu\mu} + T_{\mu\nu\alpha} + T_{\alpha\nu\mu}\right)$$

Solve for $S_{\alpha\mu\nu} = S_{\alpha\nu\mu}$

$$g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha} = \underbrace{S_{\alpha\mu\nu}}_{\to 2\Gamma_{\alpha\mu\nu} - T_{\alpha\mu\nu}} + T_{\mu\nu\alpha} + T_{\nu\mu\alpha},$$

So, $$\Gamma^{\beta}{}_{\mu\nu} = \underbrace{\frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)} + \frac{1}{2}\left(T^{\beta}{}_{\mu\nu} + T_{\mu}{}^{\beta}{}_{\nu} + T_{\nu}{}^{\beta}{}_{\mu}\right)$$

$\to$ denoted $\left\{ {\beta \atop \mu\nu} \right\}$ = Christoffel symbols.

As claimed, we have Γ in terms of g and the torsion, for a metric connection. If the torsion vanishes, then

$$\Gamma^{\beta}{}_{\mu\nu} = \left\{ {\beta \atop \mu\nu} \right\}.$$

The connection that satisfies this is a special metric connection, called the Levi-Civita connection. In a sense it is the simplest metric connection.

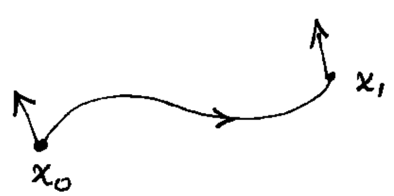
Under the Levi-Civita connection, a connection geodesic is the same as a metric geodesic, i.e.,

$$0 = \delta \int ds = \delta \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}\, ds$$

$$\Rightarrow \quad \frac{d^2 x^\mu}{ds^2} + \left\{ {\mu \atop \alpha\beta} \right\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \qquad \text{(Standard calc'n in G.R.)}$$

Now we take up curvature and holonomy.

Consider the parallel transport of a vector along a curve between x_0 and x_1.

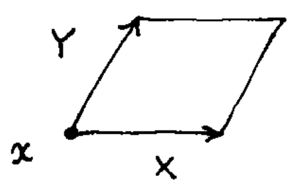


The parallel transport gives a map $\tau : T_{x_0}M \to T_{x_1}M$ (linear). In particular, if $x_1 = x_0$ (closed loop), then we have a linear map, dependent on the curve c based at x_0, $P_c : T_{x_0}M \to T_{x_0}M$.

P_c is called the holonomy of the loop c.

Notice in general $P_c \in GL(n,\mathbb{R})$, but if a metric exists and a metric connection is employed, then P_c preserves scalar products, i.e., $P_c : T_xM \to T_xM$ is an <u>orthogonal transformation</u> (a member of $SO(n)$ for an orientable, Riemannian manifold, or $SO(n,m)$ for an oriented, pseudo-Riem. manifold). In general, the set of all possible holonomies of all possible loops based at x_0 is a subgroup of $GL(n,\mathbb{R})$, called the <u>holonomy group at x_0</u>, denoted $H(x_0)$. Like the fundamental group, elements of the holonomy group depend on the loop, but they are not invariant under continuous deformation. If points x_0 and x_1 can be connected by a curve as above, then $H(x_0)$ and $H(x_1)$ are conjugate groups, $H(x_0) = \tau H(x_1) \tau^{-1}$. As abstract groups they are the same. Then one can speak of the holonomy group of the manifold. For example, the holonomy group of the 2-sphere (under the Levi-Civita connection and the obvious metric) is $SO(2)$.

If the loop is <u>infinitesimal</u> then we get an infinitesimal element of the holonomy group, i.e., an element of the <u>Lie algebra</u>. E.g., consider an infinitesimal parallelogram defined by vectors X and Y:



Then the Lie algebra element you get upon parallel transporting around the small loop depends on the area element (it is linear and antisymmetric in X, Y), i.e., it is a <u>Lie algebra-valued 2-form</u>:

$$R : \underbrace{T_xM \times T_xM}_{\text{antisumm.}} \longrightarrow \text{Lie algebra, e.g., of } H(x) \quad \mathfrak{gl}(n,\mathbb{R}) \;\; so(n).$$

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Conventions for attaching indices to R. Let the Lie algebra element be represented by an $n\times n$ matrix, in some basis in $T_x M$. Then write,

conventional minus sign

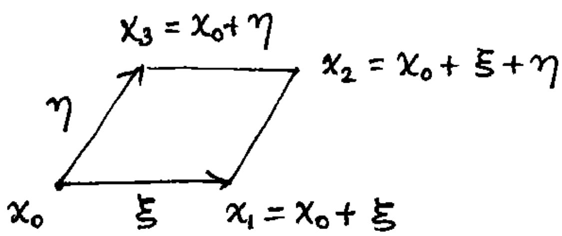
$$Z'^{\mu} = \left[\delta^{\mu}_{\nu} - R(X,Y)^{\mu}_{\cdot\nu}\right] Z^{\mu}$$

for the parallel transport of Z around the $X-Y$ parallelogram (along X first, then Y). The correction term is linear and antisymmetric in X, Y, hence

$$R(X,Y)^{\mu}_{\cdot\nu} = \underbrace{R^{\mu}{}_{\nu\alpha\beta}}_{\text{curvature tensor.}} X^{\alpha} Y^{\beta}$$

where $R^{\mu}_{\cdot\nu\alpha\beta} = -R^{\mu}_{\cdot\nu\beta\alpha}$.

How to calculate $R^{\mu}_{\cdot\nu\alpha\beta}$ in a coordinate basis $e_{\mu} = \frac{\partial}{\partial x^{\mu}}$. Change notation slightly, write ξ, η instead of X, Y (ξ, η are infinitesimals). These define an infinitesimal parallelogram in the given coordinates,



The sides of the parallelogram are straight lines in the given coordinates. Thus, on transporting a vector Z along the first leg $x_0 \to x_1$, we create a curve parametrized by t, $x^{\mu}(t) = x_0^{\mu} + t\,\xi^{\mu}$, $0 \le t \le 1$.

Notation: Let $(\xi\cdot\Gamma)$ be the $n\times n$ matrix with components,

or Γ_{ξ}

$$\underbrace{(\xi\cdot\Gamma)^{\mu}_{\cdot\nu}}_{(\Gamma_{\xi})^{\mu}_{\cdot\nu}} = \xi^{\alpha}\,\Gamma^{\mu}_{\alpha\nu}.$$

Equ. of parallel transport is

$$\frac{dZ^\mu}{dt} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}Z^\beta .$$

But $x^\alpha(t) = x_0^\alpha + t\,\xi^\alpha$

$$\frac{dx^\alpha}{dt} = \xi^\alpha$$

so $$\frac{dZ^\mu}{dt} = -(\Gamma_\xi)^\mu{}_{\cdot\beta}\, Z^\beta,$$

↳ eval. at $x(t)$.

or $$\frac{dZ}{dt} = -\Gamma_\xi(x_0 + t\xi)\,Z \qquad \text{for shord.} \quad = Z'$$

then $$\frac{d^2Z}{dt^2} = -\xi\cdot\nabla\Gamma_\xi\, Z - \Gamma_\xi\frac{dZ}{dt} = Z''.$$

$$= \left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)Z$$

where $\xi\cdot\nabla\Gamma_\xi = \xi^\mu(\Gamma_\xi)_{,\mu}$.

So,
$$Z_0' = -\Gamma_\xi\, Z_0$$
$$Z_0'' = \left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)Z_0 .$$

So, $$Z_1 = \left[\mathrm{Id} - \Gamma_\xi + \tfrac{1}{2}\left(-\xi\cdot\nabla\Gamma_\xi + \Gamma_\xi^2\right)\right]Z_0$$

Taylor series at $t=1$.

everything in [ ] eval at x_0. Z_1 = value of Z, parallel transported from x_0 to x_1. To transport ~~to~~ $x_1 \to x_2$, replace $Z_0 \to Z_1 \to Z_2$, $\xi \to \eta$, $x_0 \to x_1 = x_0 + \xi$. Thus,

$$Z_2 = \Big[\mathrm{Id} \underbrace{- \Gamma_\eta(x_0+\xi)}_{-\Gamma_\eta(x_0)\, -\, \xi\cdot\nabla\Gamma_\eta} + \tfrac{1}{2}\left(-\eta\cdot\nabla\Gamma_\eta + \Gamma_\eta^2\right)\Big]Z_1$$