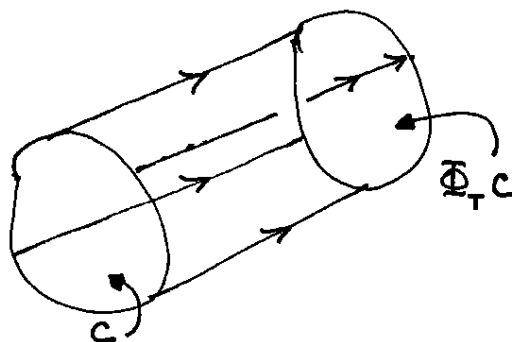


Earlier we promised a geometrical interpretation of the Cartan formula,

$$L_X = i_X d + d i_X$$

(when acting on forms). Our proof of this formula was nongeometrical (in a HW). It turns out that the Cartan formula is closely related to the proof of the Poincaré lemma.

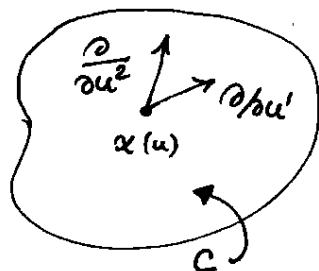
Let  $\omega \in \Omega^r(M)$ ,  $c \in C_r(M)$ . We will draw the chain  $c$  as if it is a piece of an  $r$ -dimensional surface, but everything goes through for actual chains. Let  $X \in \mathfrak{X}(M)$  be a vector field on  $M$  with advance map  $\Phi_t$ . Let the chain  $c$  flow with the flow, sweeping out an  $(r+1)$ -chain, like a "tube". After time  $t=T$ :



We can integrate  $d\omega$  over the tube and use Stokes' theorem:

$$\int_{\text{tube}} d\omega = \int_{\Phi_T c} \omega - \int_c \omega + \int_{\text{walls}} \omega.$$

Let  $(u^1, \dots, u^r)$  be coordinates on  $c$ ,



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so that

$$\int_c \omega = \int du^1 \dots du^r \omega \Big|_{x(u)} \left( \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^r} \right).$$

where  $x(u)$  ~~is~~ <sup>is</sup> the mapping:  $\mathbb{R}^r \rightarrow M$  giving coordinates on  $c$  (or, rather, the mapping defining the  $r$ -chain). As for the  $(r+1)$ -chain "tube" swept out by  $c$ , coordinates on it are  $(t, u^1, \dots, u^r)$ , where  $0 \leq t \leq T$ .

~~These~~ These coordinates correspond to a point

$$(t, u^1, \dots, u^r) \mapsto \Phi_t x(u).$$

so,

$$\int_{\text{tube}} d\omega = \int_0^T dt \int du^1 \dots du^r \underbrace{(d\omega) \Big|_{\Phi_t x(u)} \left( \mathbb{X}, \Phi_{t*} \frac{\partial}{\partial u^1}, \dots, \Phi_{t*} \frac{\partial}{\partial u^r} \right)}_{\text{---}}$$

$$\rightarrow = (i_x d\omega) \Big|_{\Phi_t x(u)} \left( \Phi_{t*} \frac{\partial}{\partial u^1}, \dots, \Phi_{t*} \frac{\partial}{\partial u^r} \right)$$

$$= (\Phi_t^* i_x d\omega) \Big|_{x(u)} \left( \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^r} \right).$$

so,

$$\int_{\text{tube}} d\omega = \int_0^T dt \int_c \Phi_t^* i_x d\omega.$$

Similarly, let  $(v^1, \dots, v^{r-1})$  be coordinates for  $\partial c$ , so that  $(t, v^1, \dots, v^{r-1})$  are coordinates for the walls. Then

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$$\int_{\text{walls}} \omega = - \int_0^T dt \int_{\partial C} \Phi_t^* i_x \omega = - \int_0^T dt \int_C d \Phi_t^* i_x \omega.$$

The minus sign takes some attention to orientation rules.

Finally,

$$\int_{\Phi_T C} \omega = \int_C \Phi_T^* \omega,$$

using property of integrals under maps. altogether, we have

$$\int_0^T dt \int_C \Phi_t^* i_x d\omega = \int_C \Phi_T^* \omega - \int_C \omega - \int_0^T dt \int_C d \Phi_t^* i_x \omega.$$

Now apply  $\frac{d}{dT}|_{T=0}$ , use  $\frac{d}{dT}|_{T=0} \Phi_T^* \omega = L_x \omega$ ,  $\Phi_0^* = \text{id}$ .

Find,

$$\int_C i_x d\omega = \int_C L_x \omega - 0 - \int_C d i_x \omega,$$

or, since  $C$  is arbitrary,

$$L_x \omega = i_x d\omega + d i_x \omega.$$

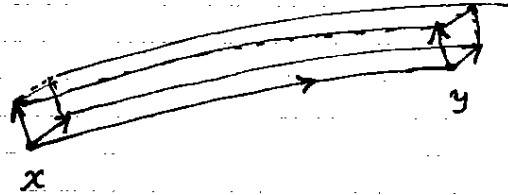
The Cartan formula arises when surfaces or chains are allowed to flow under some advance map of a vector field. It arises when we apply  $\frac{d}{dt}$  to the integral of  $\omega$  over a chain being transported by flow.

Like a small disk.



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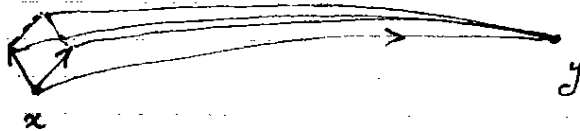
Let's apply the boxed formula in another extreme. Suppose the surface  $c$  is ~~is~~ a small parallelepiped, spanned by  $r$  vectors at a point  $x \in M$ . This surface is allowed to flow under  $\Phi_T$  to a point  $y$ .



Allowing the small parallelogram to flow means the vectors are mapped by  $\Phi_{t,x}$ . Suppose also that  $d\omega = 0$ . Then boxed formula implies,

$$\int_{\Phi_T c} \omega = \int_c \omega + \int_0^T dt \int_c d \Phi_t^* i_x \omega.$$

Now suppose the flow generates a deformation retract to the point  $y$ . Then it looks like this:



and all transported vectors collapse to 0 when they reach  $y$ . Thus

$$\int_{\Phi_T c} \omega = 0, \quad \text{and} \quad \int_c \omega = -d \int_0^T dt \int_c \Phi_t^* i_x \omega.$$

But since  $c$  is arbitrary, this implies

$$\omega = -d \int_0^T dt \Phi_t^* i_x \omega$$

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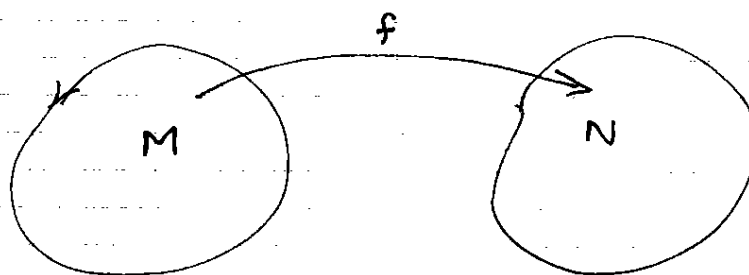
Thus we see that on a contractible region, every closed form is exact (another version of the Poincaré lemma). If you make the contraction flow along radial lines in some coordinates,

$$\frac{dx^\mu}{dt} = -x^\mu, \quad x^\mu(t) = x_0^\mu e^{-t},$$

then (with  $T \rightarrow \infty$ ) the result above implies the Volterra formula.

Note: since  $\mathbb{R}^n$  is contractible, closed  $\Leftrightarrow$  exact on  $\mathbb{R}^n$ .

A third application of the boxed formula concerns the behavior of cohomology groups under maps. Let  $f: M \rightarrow N$  be a smooth map.



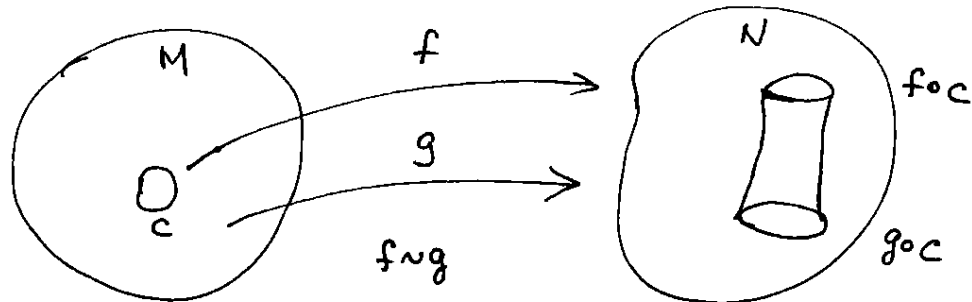
We know  $f^*$  can be used to pull-back forms on  $N$  to forms on  $M$ . Can this also be used to pull back ~~the~~ elements of cohomology groups? Let  $\omega \in \Omega^r(N)$ ,  $d\omega = 0$ . Then  $[\omega] \in H^r(N)$ . The obvious definition is

$$f^*[\omega] = [f^*\omega],$$

but we have to check that this makes sense. First,  $df^*\omega = f^*d\omega = 0$ , so  $f^*\omega$  is closed on  $M$ . Next, if  $\omega' = \omega + d\psi$ , then  $f^*\omega' = f^*\omega + f^*d\psi = f^*\omega + d(f^*\psi)$ . So,  $[f^*\omega] = [f^*\omega']$ , and the answer (the pull-back of  $[\omega]$ ) does not depend on which representative element we choose in  $[\omega]$ .

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Now let  $f, g$  be two maps:  $M \rightarrow N$ . Suppose they are homotopic.<sup>(f ~ g)</sup>  
 (We explore some connections between homotopy and cohomology.). Let  $c$  be an  $r$ -chain in  $M$ , gives 2  $r$ -chains  $f \circ c$  and  $g \circ c$  in  $N$ , that can be deformed into one another by the homotopy that deforms  $f$  into  $g$ .



Now let  $[\omega] \in H^r(N)$ , so  $d\omega = 0$ , and integrate  $d\omega$  over the volume of the tube in  $N$ , using the boxed formula. Assume that the ~~flow~~ deformation given by the homotopy ~~can be generated by~~ is the advance map of a flow associated with a vector field  $X$  on  $N$ . Then we find (replacing  $c$  in the boxed formula with  $f \circ c$ )

$$0 = \int_{g \circ c} \omega - \int_{f \circ c} \omega - d \int_{f \circ c} \int_0^1 dt \Phi_t^* i_X \omega$$

$$\text{or } 0 = \int_c g^* \omega - \int_c f^* \omega - \int_c d \int_0^1 dt f^* \Phi_t^* i_X \omega$$

or since  $c$  is arbitrary,

$$g^* \omega = f^* \omega + d\psi, \quad \psi = - \int_0^1 dt f^* \Phi_t^* i_X \omega$$

$$\text{or } [g^* \omega] = [f^* \omega]$$

$$\text{or } g^* [\omega] = f^* [\omega].$$

⑦  
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So pull-backs of cohomology groups under homotopic maps are identical,

$$f^* H^r(N) = g^* H^r(N) \quad \text{if } f \sim g.$$

In the applications above of the boxed formula, p. 3, it was assumed that the deformation associated with the homotopy could be realized as an advance map of some vector field. This was done only for reasons of laziness. You can fix this up, and the answers still hold.

Nakahara discusses "Poincaré duality," but his arguments cannot be understood on the basis of material covered so far in the course. So I will just quote the result. Let  $M$  be compact, so the  $H^r(M)$  are finite dimensional. Let  $\dim M = m$ . Then

$$\dim H^r(M) = \dim H^{m-r}(M),$$

$$\text{or } b_r(M) = b_{m-r}(M) \quad (\text{Betti numbers}).$$

The proof of this requires harmonic forms, which we consider later.

He also discusses the Künneth formula, which concerns the cohomology of Cartesian product spaces. We'll come back to this later if we need it.

Now we turn to Riemannian manifolds. These are manifolds that possess a metric tensor. We have already discussed metric tensors on a vector space  $V$ ; now we promote this idea into a field, by identifying the former  $V$  with  $T_x M$  (one metric in each tangent space). The result is a type  $(0,2)$  tensor  $g$ ,

$$g|_x: T_x M \times T_x M \rightarrow \mathbb{R} \quad (\text{at a point})$$

$$g: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{F}(M) \quad (\text{as a field}).$$

At each point  $x \in M$ ,  $g$  satisfies the requirements of a metric:

$$(1) \quad g(X, Y) = g(Y, X) \quad (\text{symmetric})$$

$$(2) \quad g \text{ is nonsingular.}$$

Property (2) can be expressed in terms of the component matrix  $g_{\mu\nu}$  of  $g$  w.r.t. some basis  $\{e_\mu\}$ ,

$$(2') \quad \det(g_{\mu\nu}) \neq 0,$$

a stmt independent of basis. By an orthogonal change of basis,  $g_{\mu\nu}$  can be diagonalized. The eigenvalues are not invariant under change of basis (which generally need not be orthogonal), but their signs are. In fact, by scaling the basis vectors by a factor  $a \neq 0$ , after  $g_{\mu\nu}$  has been diagonalized, the eigenvalues are scaled by  $a^2 > 0$ . Thus they can be scaled to  $\pm 1$  ( $0$  is excluded since  $g_{\mu\nu}$  is nonsingular). The number of  $+1$ 's and  $-1$ 's in this final form is an invariant property of  $g_{\mu\nu}$ . ~~Of all~~ The list of these numbers is the signature of  $g$ .

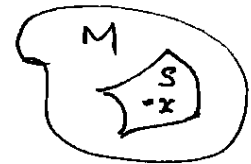


If the signature contains only +1's, then  $g_{\mu\nu}$  is positive definite, and  $M$  is said to be a Riemannian manifold. If some are +1 and others -1, then  $M$  is pseudo-Riemannian. The signature of  $g$  cannot change as we move around on  $M$  because eigenvalues are not allowed to pass through 0.

Let  $S$  be a submanifold of  $M$ , a (pseudo)-Riemannian manifold. Then  $g$  (on  $M$ ) can be restricted to  $S$ , creating a type  $(0,2)$  tensor on  $S$ . (In fact, any purely covariant tensor, a 2-form for example, can be restricted to submanifolds in the same manner.)

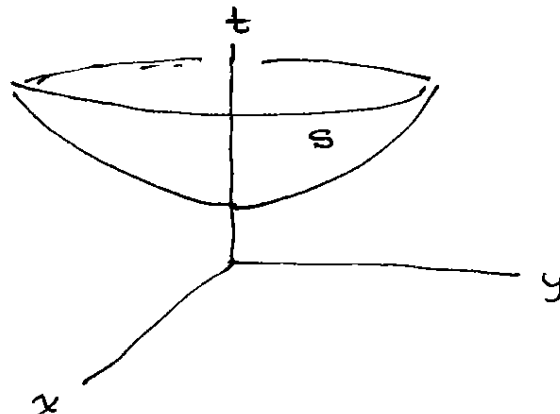
We just define

$$g|_S(x, Y) = g(x, Y),$$



where  $X, Y \in T_x S$ , are reinterpreted as elements of  $T_x M$ .  $g|_S$  is then a tensor field on  $S$ . If  $g$  is positive definite (Riemannian case), then  $g|_S$  is also, and  $S$  becomes a Riemannian manifold (every submanifold is Riemannian). But if  $M$  is pseudo-Riemannian, then  $g|_S$  ~~is~~ is not necessarily nonsingular everywhere.

Example: Let  $M = \mathbb{R}^4$  with Minkowski metric, let  $S =$  unit hyperbola (or mass shell):



Metric on  $M$ :

$$-dt^2 + dx^2 + dy^2 + dz^2.$$

Metric on  $S$ : Turns out to be Riemannian (pos. def.), but not flat,  $S$  is surface of const. negative curvature (Lobachevskian plane).

Let  $\{x^\mu\}$  be coordinates. Then in the coordinate basis,  $g$  is

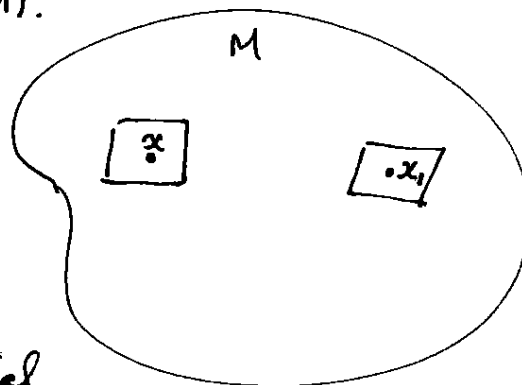
$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \quad \text{often written } g_{\mu\nu} dx^\mu dx^\nu$$

for short.

Now we turn to connections. This does not have to be discussed in the same breath with metrics, in fact, the idea of a connection is a separate geometrical concept from that of a metric, but the two subjects do have some overlap.

In general, on a manifold there is no natural way to identify two tangent spaces at two different points  $x, x_1$ , even though both are  $n$ -dimension ( $n = \dim M$ ).

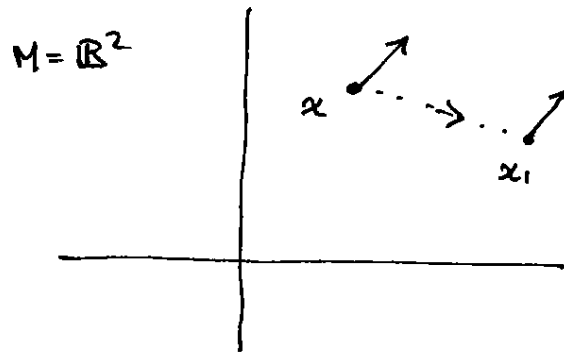
To create such an identification, it is necessary to introduce some



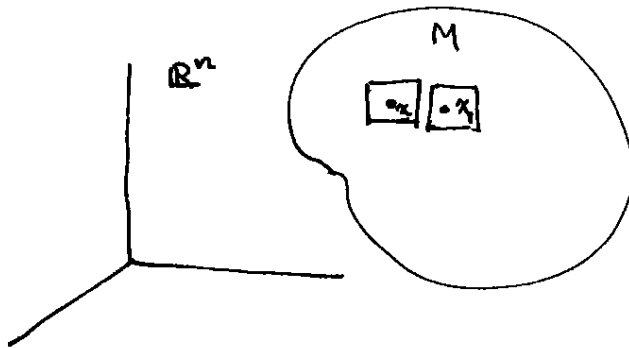
additional geometrical

structure. An exception is the case that  $M =$  a vector space  $= \mathbb{R}^n$ , then all tangent ~~vectors~~ <sup>spaces</sup> can be identified with each other (and with  $M = \mathbb{R}^n$  itself.) You just use the vector space structure to move a vector based at one point parallel to itself over to another point.

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Another example to think about is the case of a submanifold  $M$  ( $\dim M = m$ ) of Euclidean  $\mathbb{R}^n$  ( $n \geq m$ ). For now we speak intuitively (use infinitesimals, etc.)



In this case you can use the geometry (vector space + Euclidean) of  $\mathbb{R}^n$  to create an isomorphism between tangent space at nearby (infinitesimally separated) points  $x$  and  $x_1$  on  $M$ . The rule is the following. ①. Take a vector  $Y \in T_x M$  and move it parallel to itself (in  $\mathbb{R}^n$ ) over to the nearby point  $x_1 = x + \Delta x$ . Unfortunately, this transported vector is not tangent to  $M$  at  $x_1$  (it belongs to  $T_x \mathbb{R}^n$ , but not  $T_{x_1} M$ ). To fix this, ② use the metric in  $\mathbb{R}^n$  to project the vector onto the tangent plane at  $x_1$ , producing  $Y' \in T_{x_1} M$ .

You could use this <sup>construction</sup> definition to identify tangent spaces to  $M$  at remote points, but it would not be a useful definition. Instead, it's best to identify <sup>distant</sup> tangent spaces by linking them by a chain of infinitesimal increments. Then you find that the identification is

path-dependent. So for now concentrate on the identification <sup>4/1/04</sup> of nearby tangent spaces only.

In general, we want a map,

$$: T_x M \rightarrow T_{x+\Delta x} M : Y \mapsto Y', \text{ say, when } \Delta x \text{ small,}$$

such that  $Y'$  is linear in  $Y$ . Notice that we cannot say, we want the mapping to be close to the identity when  $\Delta x$  is small, since we have no way of defining an identity map between  $T_x M$  and  $T_{x+\Delta x} M$ . But if we impose coordinates  $\{x^\mu\}$  and use the coordinate basis  $\{\partial/\partial x^\mu\}$  to compute components, then the components  $Y'^\mu$  should be close to ~~those of~~  $Y^\mu$ , and the correction should be linear in  $\Delta x$ . Thus we want

$$\begin{aligned} Y'^\mu &= Y^\mu + (\text{something linear in } \Delta x, Y) \\ &= Y^\mu - \Gamma_{\alpha\beta}^\mu \Delta x^\alpha Y^\beta. \end{aligned}$$

The minus sign is conventional.  $\Gamma_{\alpha\beta}^\mu$  are the connection coefficients, they are the coefficients of the linear relationship.

In a sense,  $Y'$  is the "same" vector at  $x+\Delta x$  as  $Y$  was at  $x$ . More precisely,  $Y'$  is the parallel transported version of  $Y$ . Here we have just transported a specific vector  $Y$  at the point  $x$  over to a nearby point. But if  $Y$  is a vector field, then it has a value at  $x+\Delta x$ , and in general,  $Y(x+\Delta x) \neq Y'$ . But since

$$Y^\mu(x+\Delta x) = Y^\mu(x) + \Delta x^\alpha Y_{,\alpha}^\mu$$

↑ called simply  $Y$  above.

then the difference is

$$Y^\mu(x+\Delta x) - Y'^\mu = \Delta x^\alpha (Y_{,\alpha}^\mu + \Gamma_{\alpha\beta}^\mu Y^\beta).$$

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Now replace  $\Delta x^\alpha$  by  $\epsilon X^\alpha$ , where  $X$  is some vector in  $T_x M$  (representing the displacement), and define the covariant derivative of  $Y$  along  $X$  to be

$$\begin{aligned} (\nabla_X Y)^\mu &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [Y^\mu(x + \epsilon X) - Y^\mu] \\ &= X^\alpha (Y^\mu_{,\alpha} + \Gamma_{\alpha\beta}^\mu Y^\beta). \end{aligned}$$

In this construction,  $Y$  must be a vector field, while  $X$  need only be defined at one point  $x$  (of course, it may be a field, too).

The covariant derivative gives a notion of the directional derivative of a vector field  $Y$  along  $X$ , although it depends on a connection. The Lie derivative does not do this, since the Lie derivative  $\mathcal{L}_X Y$  involves derivatives of  $X$  as well as  $Y$ .

Now we make an abstract approach to covariant derivatives.

We wish to define,

$$\nabla: T_x M \times \mathcal{X}(M) \rightarrow T_x M \quad (X \text{ a vector at a pt})$$

$$\text{or } \nabla: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M) \quad (X \text{ a vector field})$$

such that:

$$1. \quad \nabla_{fX} Y = f \nabla_X Y$$

$$2. \quad \nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$$

$$3. \quad \nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$$

$$4. \quad \nabla_X (fY) = (Xf)Y + f \nabla_X Y$$

these properties are all satisfied by our definition above.

Put this into a basis. For now use only a coordinate basis, so that  $e_\mu = \partial/\partial x^\mu$ . Define connection components by

$$\nabla e_\alpha \equiv \nabla_\alpha$$

$$\nabla_\alpha e_\beta = \text{a vector} = e_\gamma \Gamma_{\alpha\beta}^\gamma.$$

Now use rules above to compute  $\nabla_X Y$  in components. Write  $X = X^\mu e_\mu$ ,  $Y = Y^\nu e_\nu$ . Then

$$\begin{aligned} \nabla_X Y &= \nabla_{X^\mu e_\mu} (Y^\nu e_\nu) \\ &= X^\mu \nabla_\mu (Y^\nu e_\nu) \\ &= X^\mu [(\nabla_\mu Y^\nu) e_\nu + Y^\nu \nabla_\mu e_\nu] \\ &= X^\mu [Y^\nu_{,\mu} e_\nu + Y^\nu e_\alpha \Gamma_{\mu\nu}^\alpha] \\ &= X^\mu [Y^\alpha_{,\mu} + Y^\nu \Gamma_{\mu\nu}^\alpha] e_\alpha. \end{aligned}$$

Agrees with earlier calculation of  $(\nabla_X Y)^\mu$ . Shows or indicates equivalence of two points of view.

Parallel transport. Now that we know how to parallel transport a vector over an infinitesimal segment, by integration we can transport over a finite distance, along a curve. Let  $x_0$  be the beginning of a curve, let  $Y_0 \in T_{x_0} M$  be a given tangent vector at the initial point.  $Y_0$  need only be defined at one point, i.e., it need not be a field. Let  $s$  be a parameter of the curve (any parameter).