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Summary.

homology		cohomology	
chains	$C_r(M)$	$\Omega^r(M)$	forms
cycles	$Z_r(M)$	$Z^r(M)$	cocycles (closed forms)
boundaries	$B_r(M)$	$B^r(M)$	coboundaries (exact forms)
homology	$H_r(M) = \dfrac{Z_r(M)}{B_r(M)}$	$H^r(M) = \dfrac{Z^r(M)}{B^r(M)}$	de Rham cohomology.

Explained last time why it's at least plausible that $H^r(M) = H_r(M)^*$. This is de Rham's theorem. An immediate consequence is

$$b_r = r\text{-th Betti number} = \dim H_r(M) = \dim H^r(M).$$

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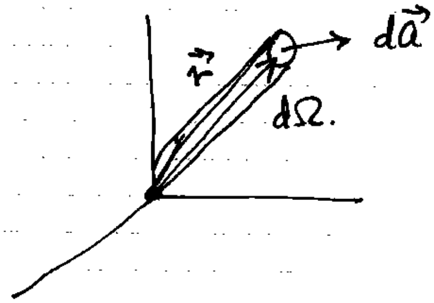
An example that will illustrate how topological information is contained in differential forms. Consider a magnetic monopole,

$$\vec{B} = g\,\frac{\hat{r}}{r^2}.$$

Magnetic fields are closely associated with 2-forms, because you integrate $\vec{B}$ over 2D surfaces to get fluxes. Let $d\vec{a}$ be an area element subtending solid angle $d\Omega$ at the origin,



Then by geometry,
$$\hat{r}\cdot d\vec{a} = r^2 d\Omega,$$

hence $\vec{B}\cdot d\vec{a} = g\,d\Omega$. To put this in the language of diff. forms, write β instead of $\vec{B}\cdot d\vec{a}$, and write $d\Omega$ in spherical coordinates:

$$\beta = g\ \sin\theta\, d\theta\wedge d\phi.$$

Since θ and ϕ are not continuous everywhere, it's not obvious that this is a smooth 2-form. So transform to Cartesian coordinates. You find

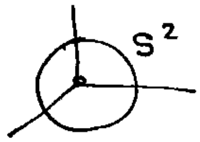
$$\beta = g\ \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{r^3},$$

which is obviously smooth everywhere except $r=0$, the location of the monopole. So to work with smooth fields, define

$$M = \mathbb{R}^3 - \{0\}.$$

Then $\beta \in \Omega^2(M)$. In fact, $d\beta = 0$, so $\beta \in Z^2(M)$.

(You can compute $d\beta$ directly.) Since $d\beta = 0$, $[\beta]$ is an element of $H^2(M)$. It is a nontrivial element (nonzero) of $H^2(M)$, because β is not <u>exact</u>. To see this, consider the integral of β over the sphere S^2 surrounding the origin:



$$\int_{S^2} \beta = 4\pi g,$$

as follows since ~~[illegible]~~ $\beta = g\, d\Omega$. But if β were exact, $\beta = d\alpha$, then $\int_{S^2} \beta = \int_{S^2} d\alpha = \int_{\partial S^2} \alpha = 0$ since S^2 is a cycle ($\partial S^2 = 0$). So β is not exact, and $[\beta] \neq 0$ defines an element of $H^2(M)$.

In ordinary language, $\beta = d\alpha$ (if it were true) would mean $\vec{B} = \nabla \times \vec{A}$. Since $\beta \neq d\alpha$, however, it means that there does not exist any $\vec{A}$ on M such that $\vec{B} = \nabla \times \vec{A}$. At least, not any <u>smooth</u> $\vec{A}$. In books it is common to introduce a "monopole string", a line on which $\vec{A}$ is singular. With such singularities, you can have $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$.

Since $[\beta] \neq 0$, we see that $H^2(M)$ is not trivial (it is not $\{0\}$). In fact, $H^2(M)$ is one-dimensional, and it is spanned by $[\beta]$. $H^2(M) \cong \mathbb{R}$, and every element of $H^2(M)$ can be written as $a[\beta]$, where $a \in \mathbb{R}$. Equivalently, if $\omega \in Z^2(M)$, then ω can be written,

$$\omega = a\beta + d\psi$$

for some $\psi \in \Omega^1(M)$. Interpreting ω as a magnetic field flux 2-form, we see that every smooth $\vec{B}$ on M is the sum of a monopole field (of some strength) plus the curl of a smooth vector potential.

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BTW, since $H^2(M) \cong \mathbb{R}$, by de Rham's thm we must have $H_2(M) = \mathbb{R}$, and there must be 2-cycles that are not boundaries. Indeed there are: S^2 is one such.

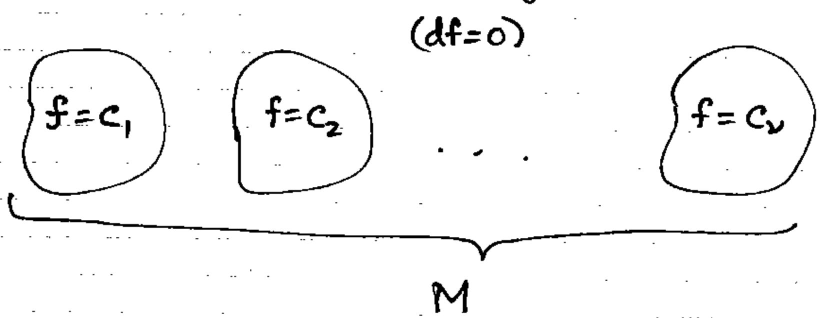
Now we develop cohomology theory and its relation to topology. First, a special case, $r=0$. We have

$$H^0(M) = \frac{Z^0(M)}{B^0(M)}.$$

0-forms on M are scalar fields, $f: M \to \mathbb{R}$. An "exact 0-form" is a scalar field f such that $f = d\alpha$ where α is a "(-1)-form". But (-1)-forms don't exist, so we understand that exact 0-forms don't exist either, except for $f=0$. That is, $B^0(M) = \{0\}$, so

$$H^0(M) = Z^0(M).$$

A closed 0-form is one satisfying $df=0$. This means $f=$ const. on each connected component of M. If M has ν connected components, then a closed 0-form on M is specified by its const. values on each component.



$\nu = \#$ connec. components.

Thus, $H^0(M) = Z^0(M) \cong \mathbb{R}^\nu$. This is the same conclusion reached earlier with homology groups, giving us an instance of de Rham's theorem.

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Now let's consider <u>bases</u> in $H_r(M)$, $H^r(M)$. Start with $H_r(M)$. Let us choose a basis in $H_r(M)$. Each basis vector is an equivalence class of ~~cl~~ r-cycles, so choose one representative element in each class, call it e_i, so that the basis ~~in~~ in $H_r(M)$ is $\{[e_i]\}$ and $e_i \in Z_r(M)$. So an arbitrary element of $H_r(M)$ can be written

$$[z] = \sum_i c_i [e_i]$$

where $z \in Z_r(M)$ and $c_i \in \mathbb{R}$ (the expansion coefficients). This means that

$$z = \sum_i c_i e_i + \partial c,$$

where $c \in C_{r+1}(M)$; thus, any cycle $z \in Z_r(M)$ can be written in this form.

Similarly for $H^r(M)$. Choose a basis in $H^r(M)$. Each basis vector is an equiv. class of closed r-forms on M. Pick representative elements, call them $\{\theta_i\}$, so that $H^r(M)$ is spanned by $\{[\theta_i]\}$, and $\theta_i \in Z^r(M)$, $d\theta_i = 0$. Now let ω be any ~~a~~ element of $Z^r(M)$, i.e. $d\omega = 0$. Then $[\omega] \in H^r(M)$ and

$$[\omega] = \sum_i a_i [\theta_i]$$

where $a_i \in \mathbb{R}$. This means

$$\boxed{\omega = \sum_i a_i \theta_i + d\psi}$$

for some $\psi \in \Omega^{r-1}(M)$. This is the general form for a closed r-form on M.

So far this is a fairly trivial statement of the defn. of a basis and of the quotient spaces $H_r(M)$ and $H^r(M)$. Now add de Rham's theorem. It tells us that the i-sums above run from 1 to $b_r(M)$ (the Betti number). It also tells us, that since $H^r(M) = H_r(M)^*$, we can choose the basis $\{[\theta_i]\}$ in $H^r(M)$ to be dual to the basis $\{[e_i]\}$ in $H_r(M)$. Let's do that, so that

$$\big([\theta_i], [e_j]\big) = (\theta_i, e_j) = \int_{e_j} \theta_i = \delta_{ij}.$$

Then the coefficients a_i in the expansion of ω above can be computed by integrating ω over the basis cycles $\{e_i\}$. To see this just substitute,

$$(\omega, e_i) = \sum_j a_j\,(\theta_j, e_i) + \cancelto{0}{(d\psi, e_i)} \quad \text{because } d\psi \text{ exact}, \ e_i \text{ a cycle}$$

$$= \sum_j a_j\,\delta_{ji} = a_i.$$

This leads to a <u>theorem:</u>

A closed form $\omega \in Z^r(M)$ is exact iff $(\omega, e_i) = \int_{e_i} \omega = 0$ for all $\{e_i\}$ in a basis in $H_r(M)$.

<u>Example of basis</u>: In monopole example above, let's take S^2 to be the basis 2-cycle, $e_1 = S^2$. Then θ_1 can be taken to be $\beta/4\pi g$, so that $(\theta_1, e_1) = 1$.

Now for formal properties of cohomology groups. Begin with the ring of differential forms on M. This is the direct sum of all forms of all possible ranks,

$$\Omega(M) = \text{ring of diff. forms on } M$$

$$= \Omega^0(M) \oplus \dots \oplus \Omega^m(M), \quad \text{where } m = \dim M.$$

An element of this ring is a (formal) linear combination of forms of different ranks, e.g., $dx \wedge dy + dz$. You can't integrate such over surfaces (that requires a single rank), but such formal linear combinations are useful nonetheless. One reason for defining such an object is to obtain a set that is closed under the exterior product $\wedge$. This is the definition of a ring, it's a set of objects that you can add or multiply, with the usual rules of distribution of addition over multiplication (and some other axioms). To say that $\Omega(M)$ is a ring mainly conveys the idea that there is a multiplication law defined, $\wedge$ in this case. A single space like $\Omega^r(M)$ is not a ring because it's not closed under $\wedge$.

Similarly we can define the cohomology ring,

$$H^*(M) = H^0(M) \oplus \dots \oplus H^m(M). = \text{cohomology ring}.$$

Here the multiplication law is again $\wedge$, but now defined on equivalence classes of closed forms (we have to make this definition). The obvious definition is (for $\omega \in Z^r(M)$, $\phi \in Z^s(M)$):

$$[\omega] \wedge [\phi] = [\omega \wedge \phi],$$

but we must check this for consistency. First, note that $\omega \wedge \phi$ is closed,

$$d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^r \omega \wedge d\phi = 0$$

since $d\omega = d\phi = 0$. Thus, $[\omega \wedge \phi]$ is meaningful as an element of $H^{r+s}(M)$. Next, must show that the def'n is indep. of the representative element in the equivalence class. Let $\omega' = \omega + d\psi$. Then

$$[\omega' \wedge \phi] = [\omega \wedge \phi + d\psi \wedge \phi].$$

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But $d\psi\wedge\phi$ is exact,

$$d(\psi\wedge\phi) = d\psi\wedge\phi + (-1)^{r-1}\overset{0}{\cancel{\psi\wedge d\phi}} \quad \text{bec. } d\phi=0.$$

so

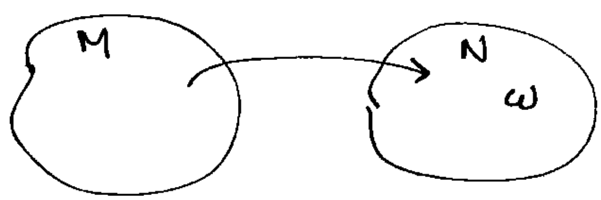
$$[\omega'\wedge\phi] = [\omega\wedge\phi + \text{exact}] = [\omega\wedge\phi].$$

Similarly if you write $\phi' = \phi + d\chi$. Altogether, we have shown that

$$\boxed{[\omega]\wedge[\eta] = [\omega\wedge\eta]}$$

is a consistent definition, and hence $H^*(M)$ is a ring under $\wedge$.

Now let's study behavior of cohomology groups and rings under maps. Let $f: M\to N$ be a smooth map between manifolds. We know how to pull back forms, i.e., if $\omega\in\Omega^r(N)$ then $f^*\omega\in\Omega^r(M)$.



$f^*\omega\in\Omega^r(M)$. $\qquad$ $\omega\in\Omega^r(N)$

What about cohomology group elements? Can we pull them back? Now let $\omega\in Z^r(M)$, $d\omega=0$, and let's try to define $f^*[\omega]$ by

$$\boxed{f^*[\omega] = [f^*\omega]}$$

(the obvious defn). But we must check consistency. First, $d(f^*\omega) = f^*(d\omega) = 0$ since $d\omega=0$, so $[f^*\omega]$ is meaningful as an element of $H^r(M)$. Next, if $\omega' = \omega + d\psi$, then

$$f^*[\omega'] = [f^*(\omega+d\psi)] = [f^*\omega + f^*d\psi] = [f^*\omega + d(f^*\psi)]$$
$$= [f^*\omega].$$

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So the result is indep. of the rep. element chosen in $[\omega]$, and the definition works. We have defined a new meaning to f^*:

$$f^* : \Omega^r(N) \to \Omega^r(M) \qquad \text{(old meaning)}$$

$$f^* : H^r(N) \to H^r(M) \qquad \text{(new meaning)}.$$

f^* is a linear map of cohomology groups.

f^* also preserves the $\wedge$ between cohomology group elements, ~~if~~ ~~we~~ as we see by pursuing the defn. of $\wedge$ and f^* on such things:

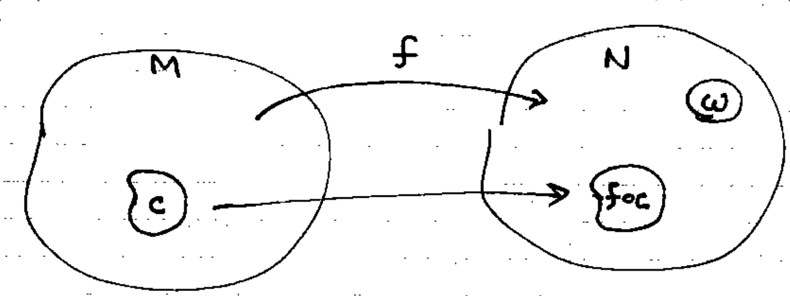
$$f^*([\omega]\wedge[\eta]) = f^*([\omega\wedge\eta]) = [f^*(\omega\wedge\eta)]$$

$$= [(f^*\omega)\wedge(f^*\eta)] = [f^*\omega]\wedge[f^*\eta]$$

$$= (f^*[\omega])\wedge(f^*[\eta]).$$

So, you can take $\wedge$ first and then apply f^*, or do it in the reverse order, answers are the same. This means that $f^* : H^*(N) \to H^*(M)$ is a <u>ring</u> <u>isomorphism</u> (another way of stating same thing).

Will need the following result concerning the behavior of integrals under maps in the subsequent discussion of potentials. Let $f: M \to N$ be a map between manifolds, let $c \in C_r(M)$ be an r-chain on M, let $\omega \in \Omega^r(N)$:



The map f can be used to push c forward to N, where it becomes

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$f \circ c$. If you think of c as a map from $\mathbb{R}^r$ to M (actually it is a linear comb. of such maps, the singular cubes or simplexes) then $f \circ c$ is a map : $\mathbb{R}^r \to N$. Then we have the following fact:

$$\boxed{\int_{f\circ c} \omega = \int_c f^* \omega.}$$

This is easily proved by resorting to the definition of the integral. This definition proceeds by pulling everything back to $\mathbb{R}^r$, so it doesn't matter if we pull it back in stages or all at once, $(f\circ c)^* = c^* \circ f^*$.

Now we turn to <u>potentials</u>. If a r form ω can be written

$$\omega = d\psi$$

for some $(r-1)$ form ψ, then we say ψ is a potential for ω. Only exact forms have (global) potentials; this is just the meaning of "exact".

The identity $dd=0$ means that exact $\Rightarrow$ closed. But closed $\not\Rightarrow$ exact, as we see from the monopole example. At least, this is true in a global sense on most manifolds. But it turns out that closed always $\Rightarrow$ exact <u>locally</u>. This is called the <u>Poincaré lemma</u> by Nakahara. This comes in several versions that we will explore.

<u>First version</u>. Let $\omega \in Z^r(M)$, $d\omega = 0$. Then ~~[illegible]~~ for every $p \in M$ there exists a neighborhood of p such that $\exists$ an $(r-1)$-form ψ such that $\omega = d\psi$ on this neighborhood.

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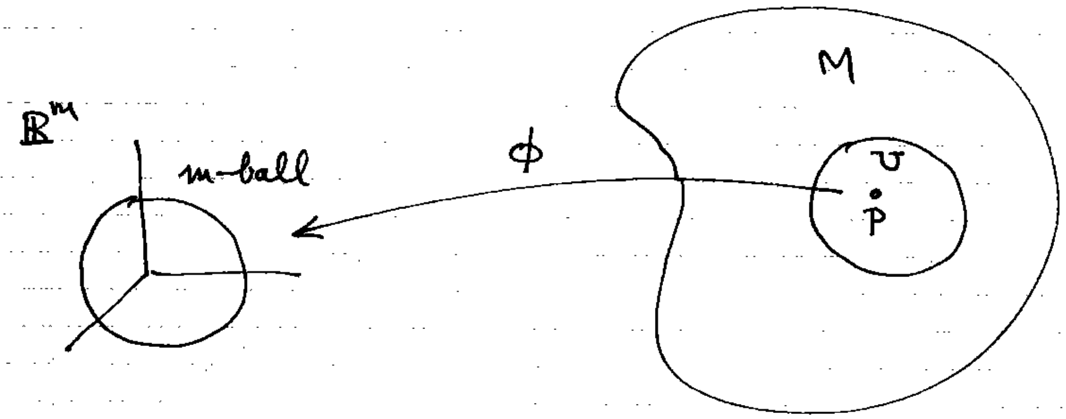
Proof. Every $p \in M$ possesses a neighborhood U (sufficiently small) that it is diffeomorphic to the m-ball (where $m = \dim M$). Take this as obvious.



Let $\phi: U \to$ m-ball be the diffeomorphism defining a coordinate chart, Let x^μ be the coordinates, and assume $x^\mu(p) = 0$. Let $\omega_{\mu_1 \ldots \mu_r}(x)$ be the components of ω in this chart. Define $\psi \in \Omega^{r-1}(M)$ by its components,

$$\boxed{\psi_{\mu_1 \ldots \mu_{r-1}}(x) = \int_0^1 dt \; t^{r-1} \; x^\sigma \; \omega_{\sigma \mu_1 \ldots \mu_{r-1}}(tx).}$$

This is the Volterra formula. From it one can prove directly that $\omega = d\psi$ inside the ball (or U). The formula is written in components, but you can see that it involves integrating ω along a straight radial path (in the given coordinates) from p to x, to get the value of ψ at x.

Proof of the Volterra formula for $r = 2$: Change notation, $\omega \to F$, $\psi \to A$ to make it look more like E+M.

Then the Volterra formula is

$$A_\mu(x) = \int_0^1 dt \ \ t \ \ x^\sigma \ F_{\sigma\mu}(tx).$$

Then calculate:

$$A_{\mu,\nu}(x) = \int_0^1 dt \left[t \ \underbrace{\delta_\nu^\sigma \ F_{\sigma\mu}(tx)}_{\to F_{\nu\mu}(tx).} + t^2 x^\sigma F_{\sigma\mu,\nu}(tx) \right]$$

$$A_{\nu,\mu}(x) = \int_0^1 dt \left[t \ F_{\mu\nu}(tx) + t^2 x^\sigma F_{\sigma\nu,\mu}(tx) \right]$$

$$(dA)_{\mu\nu}(x) = \int_0^1 dt \left[2t \ F_{\mu\nu}(tx) + t^2 x^\sigma \big(\underbrace{F_{\sigma\nu,\mu} + F_{\mu\sigma,\nu}}_{\to = -F_{\nu\mu,\sigma} = +F_{\mu\nu,\sigma} \text{ since } dF=0.} \big)(tx) \right]$$

$$= \int_0^1 dt \ \frac{d}{dt}\left[t^2 F_{\mu\nu}(tx) \right] = \quad F_{\mu\nu}(x). \qquad \text{QED}, \qquad \underline{F = dA}.$$

Next, another version of the Poincaré lemma, this time restricted to 1-forms. A basic theorem in calculus is the following. Consider the differential equations on $\mathbb{R}^n$,

$$\frac{\partial f}{\partial x^\mu} = A_\mu(x),$$

where $A_\mu(x)$ are given functions. Then this set of equations has a solution for f on a simply connected region iff $A_{\mu,\nu} = A_{\nu,\mu}$, i.e., if $dA = 0$. In other words, closed $\Rightarrow$ exact on a simply connected region of $\mathbb{R}^n$.

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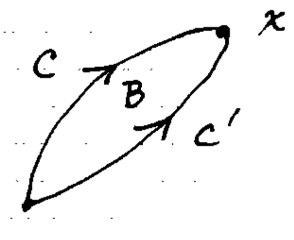
We will prove a version of this on manifolds. Let R be a simply connected region of a manifold M, let $\omega \in \Omega^1(M)$ be closed, $d\omega = 0$. Let y be a fixed pt. of R and x another such point, and let C be a curve joining y to x (confined to R). The integral $\int_C \omega$ does not depend on the path connecting y to x, because if we choose another path C', C can be continuously deformed into C' (they are homotopic, thereby sweeping out a 2-dimensional region B with boundary $\partial B = C - C'$:



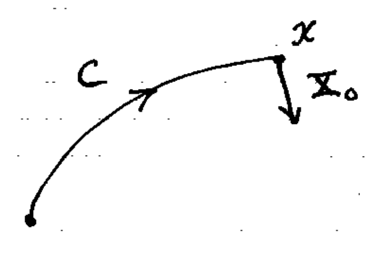
and

$$\int_C \omega - \int_{C'} \omega = \int_{\partial B} \omega = \int_B d\omega = 0.$$

So this integral defines a function

$$f(x) = \int_C \omega$$

that is, it depends only on the endpoints. Now let X_0 be a specific vector in ~~T~~ $T_x M$,



We wish to compute $(X_0 f)$.

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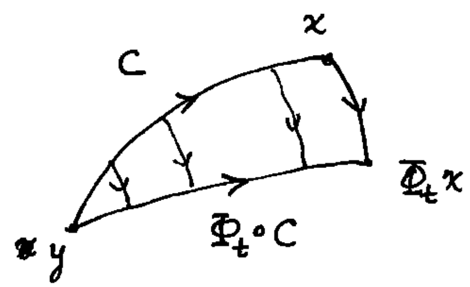
Promote $\mathbf{X}_0$ into a (smooth) vector field such that $\mathbf{X} = \mathbf{X}_0$ at x, and $\mathbf{X} = 0$ at y. Otherwise $\mathbf{X}$ is arbitrary. Let Φ_t be the advance map. Let each point of C flow with Φ_t to create a new curve $\Phi_t \circ C$:



The point y does not move under the flow since $\mathbf{X}|_y = 0$. To compute $f(\Phi_t x)$ we can use any path connecting y to $\Phi_t x$. The path $\Phi_t \circ C$ is convenient. Thus,

$$f(\Phi_t x) = \int_{\Phi_t \circ C} \omega$$

or

$$(\Phi_t^* f)(x) = \int_C \Phi_t^* \omega .$$

the Lie derivative

Now apply $\left.\frac{d}{dt}\right|_{t=0}$, use $\left.\frac{d}{dt}\right|_{t=0} \Phi_t^* = \mathcal{L}_X$ when acting on forms.

This gives

$$(\mathcal{L}_X f)(x) = \int_C \mathcal{L}_X \omega$$

use Cartan formula.

since $d\omega = 0$

or

$$(Xf)(x) = df(X)\Big|_x = \int_C \left(i_X \cancel{d}\omega + d\, i_X \omega \right)$$

$$= \int_{\partial C} i_X \omega = \omega(X)\Big|_y^x = \omega(X)\Big|_x .$$

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So, $df(x)|_x = \omega(x)|_x$, so $df(X_0) = \omega(X_0)$ so $\boxed{df = \omega}$ since x and X_0 were arbitrary. QED.

This result is interesting because it mixes homotopy and cohomology. It is discussed in different form in the book.

Actually it is very easy to understand this result from another standpoint, using the machinery discussed so far in the course. For simplicity suppose M is simply connected (instead of some region of M). Then $\pi_1(M) = \{0\}$. But we know that $H_1(M)$ is $\pi_1(M)$ divided by the commutator subgroup. But since $\pi_1(M) = \{0\}$, $H_1(M) = \{0\}$, too. This means that all closed 1-forms are exact. Note that the condition that M be simply connected is actually too strong; we could have a nontrivial $\pi_1(M)$, as long as all 1-cycles were boundaries ($H_1(M) = \{0\}$).