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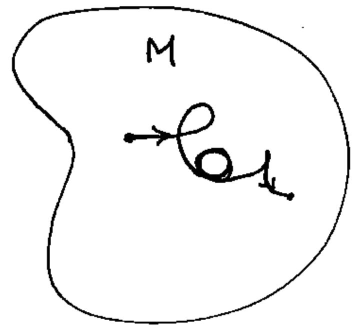
We have seen that integrating r-forms over r-dimensional submanifolds is not general enough. For example, with $r=1$, we need to integrate over paths, that is, functions



$$f: I \to M$$

$$I = [0,1] = \text{standard region} \subset \mathbb{R}$$

If $\omega = \omega_\mu(x)\, dx^\mu$ is a 1-form on M (in chart x^μ on M), then the integral we want is

$$\int_{f:I\to M} \omega = \int_0^1 dt \quad \omega_\mu(x(t))\, \frac{dx^\mu}{dt} = \int_I f^*\omega$$

where the last integral is that of a 1-form over a region of $\mathbb{R}^1$, defined previously (step 1 above). More generally, let us call a smooth map

$$\sigma : I^r \to M$$

a *singular r-cube*. $I^r \subset \mathbb{R}^r$ is the *r-cube*, a standard region in $\mathbb{R}^r$; the word "singular" is added to talk about the map σ, which need not be injective, nor need σ_* have maximal rank. For example, $\operatorname{im} \sigma$ (a subset of M) need not have dimension r, it may have self-intersections, etc. It need not be an r-dimensional submanifold of M.

Some authors (e.g. Nakahara) prefer to use a different standard region in $\mathbb{R}^r$, such as a *simplex*. Then the map is referred to as a *singular simplex*. There is no loss of generality in using cubes.

We now define the integral of an r-form $\omega \in \Omega^r(M)$ over a singular r-cube σ. It is Here $\sigma: I^r \to M$.

$$\int_\sigma \omega = \int_{I^r} \sigma^* \omega$$

which reduces the integral to the integral of an r-form over an r-dimensional region of $\mathbb{R}^r$. To put this in coordinates, let x^μ be cooordinates on M ($\mu = 1, \ldots, m = \dim M$, $m \geq r$), and let u^α, $\alpha = 1, \ldots, r$ be the standard (Euclidean) coordinates on $\mathbb{R}^r$. Then

$$\int_\sigma \omega = \int_0^1 du^1 \ldots \int_0^1 du^r \quad \omega_{\mu_1 \ldots \mu_r}(x(u)) \frac{\partial x^{\mu_1}}{\partial u^1} \ldots \frac{\partial x^{\mu_r}}{\partial u^r}.$$

The most general integral is taken over linear combinations of singular r-cubes. We consider only real coefficients here. If $\{\sigma_i^r\}$ is a set of singular r-cubes, then we define

$$c^r = \sum_i a_i \sigma_i^r, \qquad a_i \in \mathbb{R}$$

as an <u>r-chain</u>. Integrals over r-chains are computed by

$$\int_{c^r} \omega = \sum_i a_i \int_{\sigma_i^r} \omega .$$

The set of all r-chains on M is the <u>r-th chain group</u>, $C_r(M, \mathbb{R})$ (we will drop the $\mathbb{R}$, it being henceforth understood.) The r-th chain group is a group in the sense that it is a vector space (an Abelian group). This is like the simplicial chain groups

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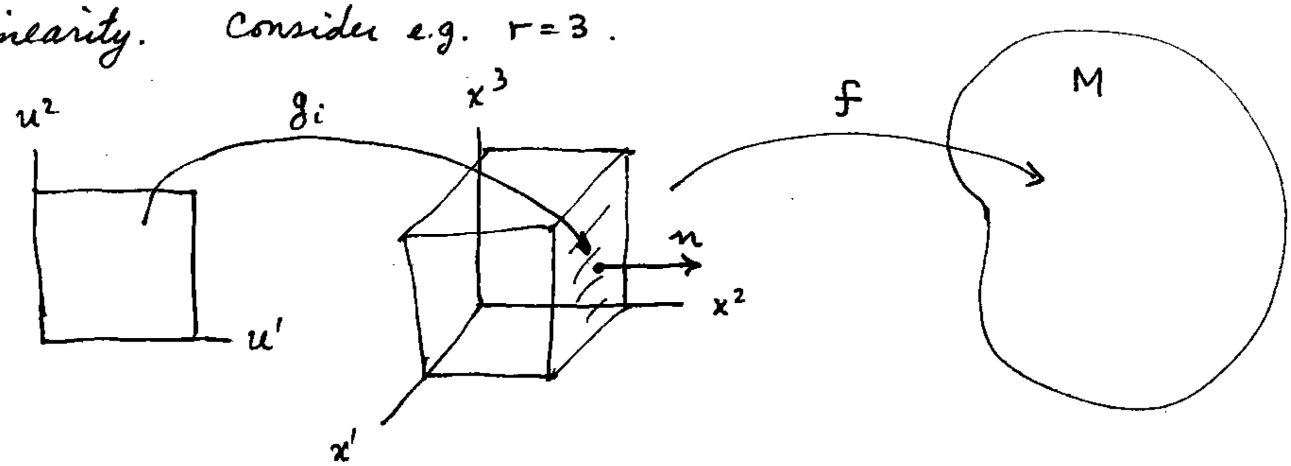
considered earlier, except now they include singular cubes, and now they are ∞-dimensional.

We now define the boundary operator ∂, when acting on singular r-cubes. Once that is defined, ∂ becomes defined on chains by linearity. Consider e.g. $r=3$.



We have 6 faces, $i=1,\ldots,6$. Each face will be associated with a singular 2-cube. But a singular 2-cube is a map from the std 2-cube (square) in $\mathbb{R}^2$ to M, and the faces of $I^3 \subset \mathbb{R}^3$ are subsets of $\mathbb{R}^3$, not $\mathbb{R}^2$. So we introduce new maps $g_i : \mathbb{R}^2 \to \mathbb{R}^3$ that map $I^2 \subset \mathbb{R}^2$ onto a face of $I^3 \subset \mathbb{R}^3$. Let x^μ be coords in $\mathbb{R}^3$ and u^α be coords in $\mathbb{R}^2$ (or (x,y,z) and (u,v)). The map g_i must assign the right orientation to the face, defined by saying that $(n, \partial/\partial u^1, \partial/\partial u^2)$ are positively oriented in $\mathbb{R}^3$, where n is an outward normal to the face, and $\partial/\partial u^1$, $\partial/\partial u^2$ span the face. For example, in the diagram above, we map I^2 onto the face of I^3 by writing,

$$\left.\begin{aligned} u &= z \\ v &= x \\ 1 &= y \end{aligned}\right\} .$$

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Then

$$f\circ g_i : I^2 \to M$$

is the i-th face, a singular 2-cube. This defines $\partial: C_r(M) \to C_{r-1}(M)$ or ∂_r, more precisely

Then we define, as in homology theory,

$$Z_r(M) = \{c \in C_r(M) \mid \partial c = 0\} = r\text{-th cycle group} = \ker \partial_r$$

$$B_r(M) = \{c \in C_r(M) \mid c = \partial b, \text{ some } b \in B_{r+1}\} = r\text{-th boundary group} \quad = \operatorname{im} \partial_{r+1}.$$

and we have $\partial^2 = 0$, as before, so $B_r(M) \subset Z_r(M)$. And we define

$$H_r(M) = \frac{Z_r(M)}{B_r(M)}. \quad r\text{-th homology group.}$$

This is same group as before.

The properties of ∂ on chains is mirrored in the properties of d acting on forms. The terminology reflects this:

$d: \Omega^r(M) \to \Omega^{r+1}(M)$ or d_r

$$\Omega^r(M) = \{ r\text{-forms on } M\}$$

closed forms $\to$ $Z^r(M) = \{\omega \in \Omega^r(M) \mid d\omega = 0\}$, r-th cocycle group. $= \ker d_r$

exact forms $\to$ $B^r(M) = \{\omega \in \Omega^r(M) \mid \omega = d\beta, \text{ some } \beta \in \Omega^{r-1}(M)\}$. $= \operatorname{im} d_{r-1}$ r-th coboundary group

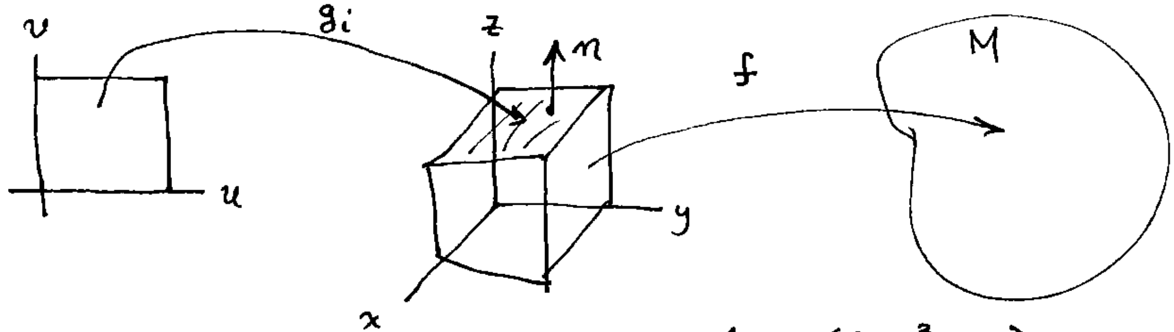
And because $d^2 = 0$, we have $B^r(M) \subset Z^r(M)$. And we define

$$H^r(M) = \frac{Z^r(M)}{B^r(M)} = \frac{\text{closed}}{\text{exact}} = \frac{\text{cocycles}}{\text{coboundaries}} = r\text{-th cohomology group}$$

To explore this association, we need Stokes' theorem, which says, if $c \subset C^{r+1}(M)$, $\omega \in \Omega^r(M)$,

$$\boxed{\int_{\partial c} \omega = \int_c d\omega}$$

To prove this it suffices to consider a single singular r-cube, since chains are lin. combs. of such things. Will do example of 3-forms. Let $\dim M = m =$ anything. Let $\omega \in \Omega^2(M)$, so $d\omega \in \Omega^3(M)$.



$c =$ chain $(f : I^3 \to M)$.

Let $\alpha = f^*\omega$, $\alpha \in \Omega^2(\mathbb{R}^3)$. This means α has 3 nonzero components,

$$\alpha = \alpha_x \overset{dy\wedge dz}{\cancel{dz\wedge dy}} + \alpha_y\, dz\wedge dx + \alpha_z\, dx\wedge dy,$$

$$d\alpha = d(f^*\omega) = f^*(d\omega)$$

$$= \left(\frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} + \frac{\partial \alpha_z}{\partial z}\right) dx\wedge dy\wedge dz$$

So,

$$\int_c d\omega = \int_{I^3} d\alpha = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left(\quad \right) = 3 \text{ terms}.$$

Look at z-term, $= \int_0^1 dx \int_0^1 dy \, [\alpha_z(x,y,1) - \alpha_z(x,y,0)]$.

We get 6 terms altogether for $\int_c d\omega$. Now consider $\int_{\partial c} \omega$.

Look at the top face of the cube: Let $g_i : I^2 \to$ top face of I^3.

$$x = u$$
$$y = v$$
$$z = 1$$

Then $(f \circ g_i)^* \omega = g_i^* f^* \omega = g_i^* \alpha$. But

$$g_i^* \alpha = \alpha_z(x,y)\, dx \wedge dy = \alpha_z(u,v)\, du \wedge dv,$$

since $dz = 0$ on top face. Thus,

$$\int_{I^2} (f \circ g_i)^* \omega = \int_{I^2} g_i^* \alpha = \int_0^1 du \int_0^1 dv \; \alpha_z(u,v,1).$$

This is one of the 6 terms from the integral $\int_c d\omega$. The other 5 add up to make $\int_c d\omega$.

Table:

homology	cohomology
$C_r(M)$	$\Omega^r(M)$
$Z_r(M)$	$Z^r(M)$
$B_r(M)$	$B^r(M)$
$H_r(M)$	$H^r(M)$.

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These spaces are dual to each other, in a certain sense. Notation, let $\omega \in \Omega^r(M)$, $c \in C_r(M)$, then write

$$\int_c \omega = (\omega, c) \in \mathbb{R}.$$

Thus r-forms are real-valued, linear operators on the space of r-chains, and vice versa. This suggests that maybe $\Omega^r(M)$ is the dual space to $C_r(M)$,

$$\Omega^r(M) = C_r(M)^*.$$

These are ∞-dimensional vector spaces, so making this interpretation precise involves a big effort. We will just proceed as if it is true.

In an earlier HW problem, had vector space V, its dual V^*, a subspace $U \subset V$, and $X^* \subset V^*$, where X^* is set of forms that annihilate U. Then we had a thm,

$$\dim U + \dim X^* = \dim V. \qquad (\ker X^* = U)$$

So if U is big (high dimensionality), X^* is small and vice versa. (You can specify a vector subspace $U \subset V$ either by vectors that span it, or by forms that annihilate it.)

So what are the forms in $\Omega^r(M)$ that annihilate $Z_r(M) \subset C_r(M)$? (note if they annihilate Z_r, they annihilate B_r, too). Answer: $B^r(M)$ (coboundaries, or exact forms). How to see: Let $\beta \in B^r(M)$, $z \subset Z_r(M)$ [β = exact, z = a cycle]. Then ($\beta = d\gamma$, some $\gamma \in \Omega^{r-1}(M)$).

$$\int_z \beta = \int_z d\gamma = \int_{\partial z} \gamma = 0.$$

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And what are the forms that annihilate $B_r(M) \subset Z_r(M) \subset C_r(M)$? Ans: $Z^r(M)$ (cocycles, or closed forms). How to see: Let $b \in B_r(M)$ (a boundary, so $b = \partial c$, some $c \in C^{r+1}(M)$), and let $\zeta \in Z^r(M)$, (a closed form, $d\zeta = 0$). Then

$$\int_b \zeta = \int_{\partial c} \zeta = \int_c d\zeta = 0.$$

Conversely, interpreting $C_r(M)$ as the operators and $\Omega^r(M)$ as the operands, then $B_r(M)$ is the space that annihilate $Z^r(M)$, and $Z_r(M)$ annihilates $B^r(M)$.

What is the space dual to $H_r(M)$ (homology group)? An element of $H_r(M)$ is $[z]$ where $z \in Z_r(M)$ is a cycle and $[z] = [z+b]$ where $b \in B_r(M)$ is a boundary. So, an operator acting on $H_r(M)$ would be one that acts on $Z_r(M)$ but annihilates boundaries, so the answer does not depend on which cycle z in $[z]$ is chosen. This means it should be a cocycle, because if $\zeta \in Z^r(M)$, then

$$(\zeta, z+b) = (\zeta, z) + \overset{0}{(\zeta, b)}.$$

So, $\zeta \in Z^r(M)$ can be associated with an element of $H_r(M)^*$. (You can think of $(\zeta, [z])$.) However, this element of $H_r(M)^*$ is not uniquely specified by ζ, because $\zeta' = \zeta + \beta$, where $\beta \in B^r(M)$, $\beta = d\gamma$, specifies the same map: $H_r(M) \to \mathbb{R}$:

$$(\zeta + \beta, z) = (\zeta, z) + \underbrace{(\beta, z)}_{\to (d\gamma, z) = 0}.$$

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Thus, the element of ~~H~~ $H_r(M)^*$ is specified by an equivalence class $[\xi] = [\xi + \beta]$, $\beta = d\gamma$, that is, an element of $H^r(M)$. This suggests that

$$H_r(M)^* = H^r(M).$$

de Rham's Theorem asserts that this is correct, and moreover that in the case M is compact, $H^r(M)$ is finite dimensional. This dimensionality is the r-th Betti number,

$$b_r = \dim H_r(M) = \dim H^r(M).$$

$H^r(M)$ is properly called the r-th de Rham cohomology group.

Remark: In Stokes thm,

$$(\omega, \partial c) = (d\omega, c)$$

we can see that d is the pull-back of ∂. That is,

$$\partial_r : C_r(M) \to C_{r-1}(M)$$

$$\partial_r^* : C_{r-1}(M)^* \to C_r(M)^*$$

$$: C^{r-1}(M) \to C^r(M)$$

Thus $d_{r-1} = \partial_r^*$. Nakahara mistakenly calls this the adjoint (which requires a metric).