(c)

3/11/04

Summary :

space	at e	Fields.	
$\mathfrak{g}$	V_μ	X_μ	Basis of vector fields
$\mathfrak{g}^*$	β^μ	θ^μ	Basis of 1-forms.

$$\left.\begin{aligned} X_\mu\big|_a &= L_{a*} V_\mu \\ \theta^\mu\big|_a &= L^*_{a^{-1}} \beta^\mu \end{aligned}\right\} \text{ left-translating to get fields}$$

$$[V_\mu, V_\nu] = C_{\mu\nu}{}^\sigma V_\sigma$$

$$[X_\mu, X_\nu] = C_{\mu\nu}{}^\sigma X_\sigma$$

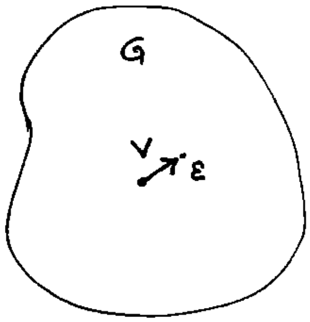
$$d\theta^\mu(X_\nu, X_\sigma) = -C_{\nu\sigma}{}^\mu = (d\theta^\mu)_{\nu\sigma},$$ components w.r.t. X_μ basis.

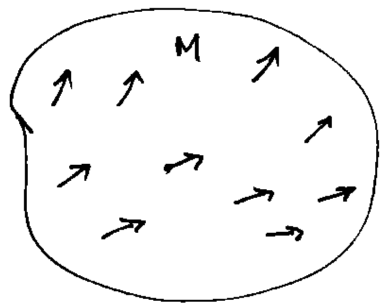
$$d\theta^\mu = -\frac{1}{2} C_{\nu\sigma}{}^\mu \, \theta^\nu \wedge \theta^\sigma$$

[special case of $\omega \in \Omega^r(M)$, $\omega = \frac{1}{r!} \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$]

3/11/04

Next we consider <u>induced vector fields</u>, which you have when you have an action of a Lie group G on a manifold M. First the intuitive picture. Consider a vector $V \in \mathfrak{g}$. Intuitively its base is the identity e and its tip is a nearby (near-identity) group element, call it ε. The map $\Phi_e = \mathrm{id}_M$ does



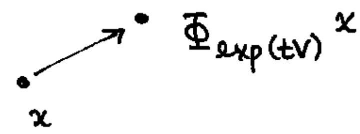


nothing to points of M, but Φ_ε causes the points of M to get up and move a short distance, creating a vector field on M. In this way we associate a single vector $V \in \mathfrak{g}$ with a vector field $V_M \in \mathfrak{X}(M)$. (V_M is denoted $V^\#$ by Nakahara.)

~~To make this more precise, replace ε by $\sigma(t) = \ldots e = \exp(tV)$ for small t, using earlier notation for integral curves on the group manifold, and consider the action of Φ~~

To make this more precise we need to talk about advance maps on the group manifold, earlier denoted $\Phi_{V,t}$, and the action of G on M, which will be denoted $g \mapsto \Phi_g$. To avoid confusion, let's use $\Psi_{V,t}$ for the advance map on G, Φ_g for the action of G on M. When t is small, $\Psi_{V,t}\, e = \exp(tV)$ is close to e, so we can identify it with ε above in the picture. When acting on $x \in M$, $\Phi_\varepsilon = \Phi_{\exp(tV)}$ causes x to move to a nearby point,

3/11/04



thereby making a small vector on M. Letting this vector act on a ~~function~~ scalar $f: M \to \mathbb{R}$, we have

$$f(\Phi_{\exp(tV)}\, x) - f(x)$$

$$= (\Phi^*_{\exp(tV)} f)(x) - f(x)$$

$$= ((\Phi^*_{\exp(tV)} - 1) f)(x). \qquad \text{(think } t \text{ small).}$$

Suggests we define $V_M \in \mathfrak{X}(M)$ by

$$(V_M f)(x) = \left(\left(\frac{d}{dt}\bigg|_{t=0} \Phi^*_{\exp(tV)} \right) f \right)(x),$$

or, since x and f are arbitrary,

$$\boxed{V_M = \frac{d}{dt}\bigg|_{t=0} \Phi^*_{\exp(tV)}}$$

(Both sides understood as operators: $\mathcal{F}(M) \to \mathcal{F}(M)$.)

See Nakahara Eq. 5.160. He drops the star on Φ and just writes g instead of Φ_g, where here $g = \exp(tV)$.

V_M is called the <u>induced vector field</u>. It is also called the <u>infinitesimal generator</u> of the action $g \mapsto \Phi_g$.

3/11/04

An equivalent way to define the induced vector field. V_M eval. at a point $x \in M$ is an equivalence class of curves. One of these curves is easy to write down. Let $c: \mathbb{R} \to M$ be defined by

$$c(t) = \Phi_{\exp(tV)}\, x.$$

Then $c(0) = x$, and $[c] = V_M|_x$.

An application of induced vector fields. Let M be the configuration space of a mechanical system. Impose a chart with coordinates q^μ. The Lagrangian is a function $L(q, \dot{q})$. Let G be a group with an action $g \mapsto \Phi_g$ on M, and suppose that L is invariant under the group action. This means that $\Phi_g^* L = L$, $\forall g \in G$. (But we won't define what Φ_g^* means here, just say that there is an obvious definition.) Then for every $V \in \mathfrak{g}$ there is a conserved quantity C_V, where

$$C_V = (p, V_M) = p_\mu (V_M)^\mu, \qquad \frac{dC_V}{dt} = 0.$$

Here $p_\mu = \dfrac{\partial L}{\partial \dot{q}^\mu}$ is the canonical momentum. This is <u>Noether's</u> <u>theorem</u>.

3/11/04

Last topic is the <u>adjoint representation</u> (confusingly called the "adjoint map" by Nakahara). This is a linear action of G on its own Lie algebra, $g \mapsto Ad_g$, where $Ad_g : \mathfrak{g} \to \mathfrak{g}$. The definition is simply ~~Adg~~

$$Ad_a = I_{a*} \quad \text{eval. at } e,$$

where $I_a = L_a R_{a^{-1}}$ (the inner automorphism). Thus, $I_{a*} = L_{a*} R_{a^{-1}*} = R_{a^{-1}*} L_{a*}$ (since left and right translations commute). When I_{a*} acts on a vector $V \in T_eG = \mathfrak{g}$, first L_{a*} maps it to a vector in T_aG, then $R_{a^{-1}*}$ maps it back to $\mathfrak{g}$. So, I_{a*} ~~action~~ maps $\mathfrak{g} \to \mathfrak{g}$. It also satisfies $I_{a*} I_{b*} = (I_a I_b)_* = I_{ab*}$, since $a \mapsto I_a$ is an action. Thus, (changing notation),

$$Ad_a Ad_b = Ad_{ab}.$$

For a matrix group, a vector $V \in \mathfrak{g}$ is represented by a matrix V, group element a is rep'd by a matrix A, and $Ad_a V$ is rep'd by the matrix AVA^{-1}. This is the adjoint rep. for matrix groups.

(Go to p. 6, 3/4/04 for notes on integrating n-forms over an n-dimensional manifold.)

We will take the following approach to integrating differential forms:

(1) Integrating an m-form over an m-dimensional region of $\mathbb{R}^m$.

(2) Integrating an m-form over an m-dimensional, orientable manifold M.

(3) Integrating an s-form over an s-dimensional orientable submanifold of M $(s \leq m)$.

(4) Integrating an r-form over an r-chain.

<u>Step 1</u>. Let $\omega \in \Omega^m(\mathbb{R}^m)$, and let R be a "nice" region of $\mathbb{R}^m$.

Then ω has only one independent component ρ, given by

$$\omega = \rho(x^1, \ldots, x^m)\, dx^1 \wedge \ldots \wedge dx^m.$$

That is, we use the standard coordinates on $\mathbb{R}^m$ to define ρ. Then we define

$$\int_R \omega = \int_R \rho(x^1, \ldots, x^m)\, dx^1 dx^2 \ldots dx^m$$

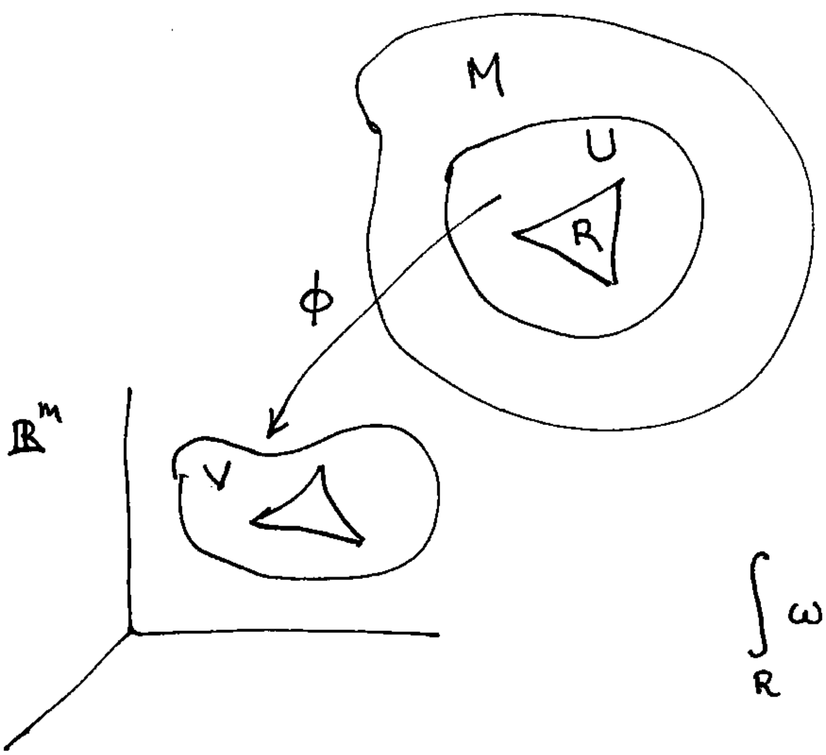
where the latter integral is an ordinary Riemann integral

(in particular, the integral does not depend on the ordering of the dx's.)

Step 2. Let M be an orientable, m-dimensional manifold, and let $\omega \in \Omega^m(M)$. We choose an oriented atlas on M, and divide M into regions $\{R_i\}$ such that each region R_i lies in one chart with coordinates x^μ. Then we define

$$\int_M \omega = \sum_i \int_{R_i} \omega ,$$

where the integral over one region R is given by

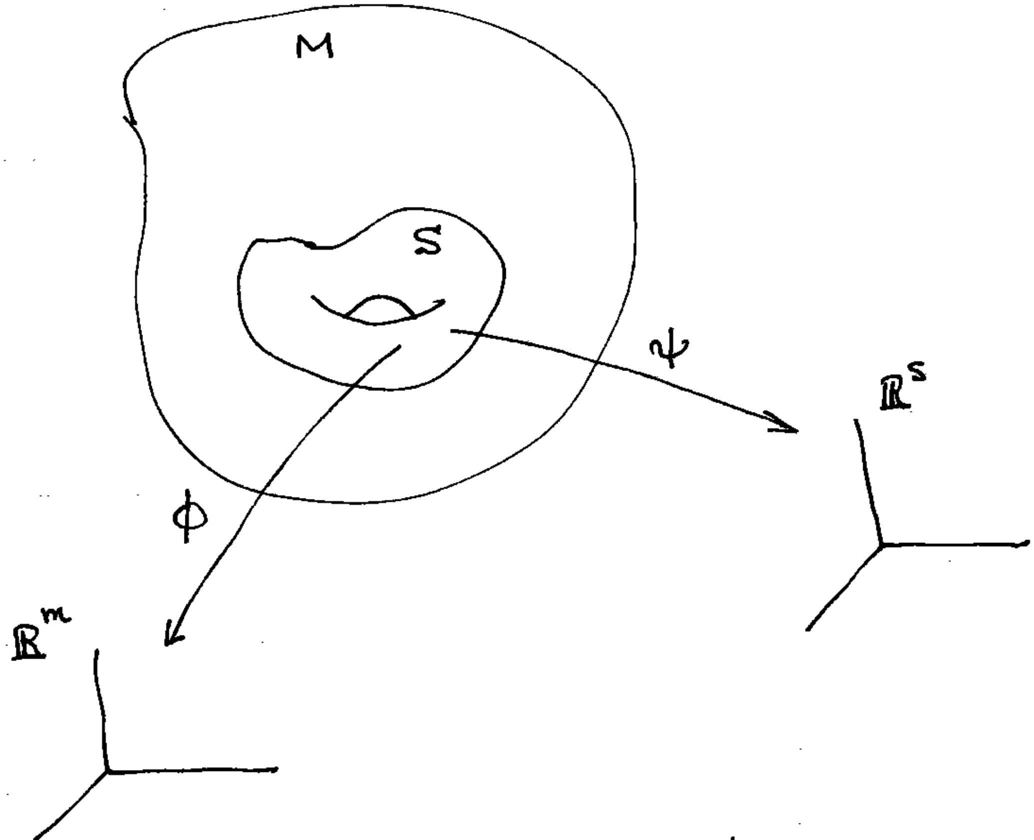


$$\int_R \omega = \int_{\phi(R)} \phi^{*-1} \omega .$$

The latter formula is the integral of an m-form over $\mathbb{R}^m$, which was defined in Step 1. (ϕ is the invertible coordinate map, $\phi: U \to V \subset \mathbb{R}^m$.)

3/11/04

Step 3. Let S be an orientable, s-dimensional submanifold of an m-dimensional manifold M. ($s \le m$). A submanifold of a manifold is a subset that is also a manifold. Let $\omega \in \Omega^s(M)$. Note that ω has $\binom{m}{s}$ indep. components in some chart.



Since S is a manifold, we impose coordinates $u^\alpha = (u^1, \dots, u^s)$ on it with some chart ψ. We let this chart have some overlap with the chart ϕ on M, with coordinates $x^\mu = (x^1, \dots, x^m)$. In ordinary language, we would say that the functions,

$$x^\mu = x^\mu(u^\alpha) = x^\mu(u^1, \dots, u^s)$$

are the "equations of the surface". They represent the map

$$\phi \circ \psi^{-1} : \left(\substack{\text{region of} \\ \mathbb{R}^s}\right) \to \mathbb{R}^m.$$

Now $\omega \in \Omega^s(M)$. But a form on a ~~space~~ manifold can always be restricted to a submanifold. In the present case, we denote the restricted form $\omega|_S$. It acts on tangent vectors to S at a point $x \in S$ by

$$(\omega|_S)|_x \,(X_1,\dots,X_s) = \omega|_x(X_1,\dots,X_s),$$

where $X_1,\dots,X_s \in T_xS \subset T_xM$. The vectors $X_1,\dots,X_s$ have a dual interpretation, as vectors tangent to M, and as tangent to S. This can also be written,

$$\omega|_S = i^*\omega,$$

where $i: S \to M$ is the <u>inclusion</u> <u>map</u> (identity map on S regarded as subset of M).

Notice that it is not possible, in general, to restrict vectors, only forms.

Finally, we define

$$\int_S \omega = \int_S \omega|_S,$$

which reduces ~~one~~ Step 3 to Step 2.

Note: the final integral gets reduced (as in Step 2) to integrals over the s-dimensional coordinate space. It is interesting to write one of these integrals out in terms of the components $\omega_{\mu_1\dots\mu_s}$ of ω on M (in chart x^μ on M).

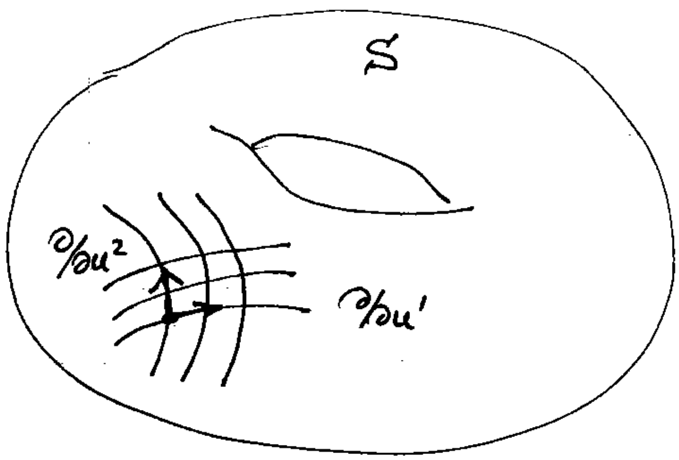
The integral has the form,

$$\int du^1 \ldots du^s \quad \underbrace{\omega_{\mu_1 \ldots \mu_s}(x(u)) \, \frac{\partial x^{\mu_1}}{\partial u^1} \cdots \frac{\partial x^{\mu_s}}{\partial u^s}} .$$

The integrand ( ) can also be written,

$$\omega\left(\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^s}\right)$$

which is ω acting on the set of basis vectors on the submanifold S induced by the coordinates u^α.



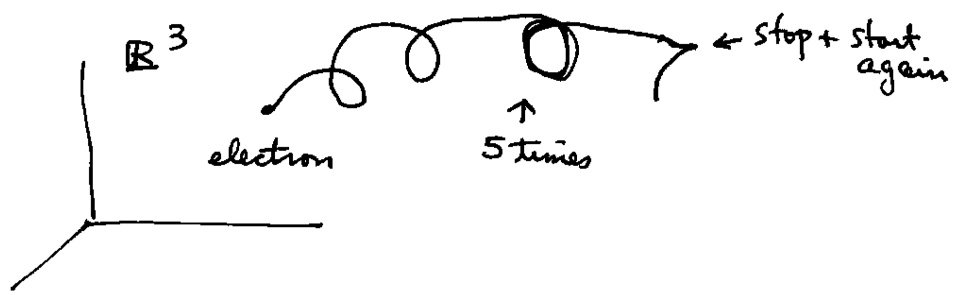
In effect, these basis vectors $\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^s}$ span and define a small s-dimensional parallelopiped on the surface S. ω acts on this parallelepiped, producing a small contribution to the integral. The integral is the sum over the small parallelepipeds.

In Step 2 and Step 3, we require M or S to be orientable, because otherwise the integral depends on the coordinates used. If M or S are orientable, then the answer does not depend on the coordinates, apart from orientation.

3/11/04

that is, two atlases of the same orientation give the same ~~sign~~ answer, one of the opposite orientation reverses the sign of the answer.

Step 4. Even in simple examples, it is easy to see that integrating over manifolds or submanifolds is not sufficient for real problems. Consider for example the work required to move an electron from one place to another in an electric field. This is a line integral (one-dimensional, using 1-forms). The path of the electron need not be a submanifold (one-dimensional). It may have self-intersections, the path may retrace or repeat itself, or the electron may stop for a while.



Obviously we want to parameterize the path by time or some other parameter, say, on the interval $I = [0,1]$. Thus the path is a map,

$$f : I \to M \qquad (= \mathbb{R}^3 \text{ for electron}).$$

and it is this map that we want to integrate over. The map f need not be injective and f_* need not have maximal rank.