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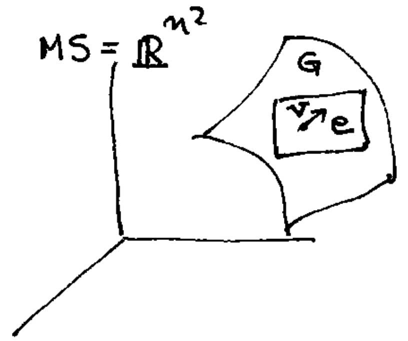
Now we consider how things like the Lie algebra, one-parameter subgroups, etc., are expressed in terms of matrices in the case that we have a matrix group. Consider a real matrix group, for simplicity. As explained previously, such a group can be thought of as a submanifold of "matrix space" $MS = \mathbb{R}^{n^2}$, where our group consists of real, $n \times n$ matrices.



Then every point of G is also a matrix, and matrix multiplication and inversion correspond to group multiplication and inversion. (and $e = I$)

We will not attempt to put coordinates on G, but coordinates on MS may be taken to be the components of a matrix. That is, if $M \in MS$, let us write

$$M = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

so that $\{x_{ij}\}$ are the coordinates on MS.

A vector at a point to G can also be interpreted as a vector at the same point to MS, and thus can be expanded as a linear combination of the basis vectors $\partial/\partial x_{ij}$. For example, if $V \in \mathfrak{g}$, then we can write

$$V = \sum_{ij} V_{ij} \frac{\partial}{\partial x_{ij}},$$

where V_{ij} is a matrix. This is the usual matrix belonging to the Lie algebra of a matrix group. For example, if $G = SO(3)$, $MS = \mathbb{R}^9$, then 3/9/04 the Lie algebra consists of antisymmetric matrices. Then, $V_{ij} = -V_{ji}$.

Then it also happens that the one-parameter subgroups $\exp(tV)$ defined in the differential-geometric setting coincide with matrix exponentiation $\exp(tV)$ (same notation). It also happens that the $[,]$ bracket on the Lie algebra becomes the ordinary matrix commutator. Other objects (left and right translations, left-invariant vector fields, etc) can also be translated into matrix language.

Return to the differential geometry of Lie groups. Let $\{V_\mu, \mu = 1,\ldots,n\}$ ($n = \dim V$) be a basis in $\mathfrak{g}$. Let $X_\mu = X_{V_\mu}$ be the corresponding LIVF's. Now the bracket $[V_\mu, V_\nu]$ is also a vector in $\mathfrak{g}$, so it can be expanded in terms of the $\{V_\mu\}$,

$$[V_\mu, V_\nu] = c_{\mu\nu}{}^\sigma V_\sigma,$$

where $c_{\mu\nu}{}^\sigma$ are the expansion coefficients. These numbers are called the <u>structure</u> <u>constants</u> of the Lie algebra, although they are not really constant, instead they are the components of a type $(1,2)$ tensor at $e \in G$. (They depend on the basis.) If we left-translate the above, we get

$$[X_\mu, X_\nu] = c_{\mu\nu}{}^\sigma X_\sigma,$$

with the same $c_{\mu\nu}{}^\sigma$ (which do not depend on position).

A different point of view results from shifting attention from <u>vector fields</u> to <u>forms</u> (the dual point of view). Let $\mathfrak{g}^* = T_e^* G$ be the dual of $\mathfrak{g}$ (the Lie algebra). Let $\{\beta^\mu, \mu = 1,\ldots,n\}$ be the basis in $\mathfrak{g}^*$ dual to $\{V_\mu, \mu = 1,\ldots,n\}$, the (some) given basis in $\mathfrak{g}$.

That is, $\beta^\mu \in \mathfrak{g}^*$,

$$\beta^\mu(V_\nu) = \delta^\mu_\nu.$$

Then define a 1-form $\theta^\mu \in \mathfrak{X}^*(G)$ by

$$\theta^\mu\big|_a = L^*_{a^{-1}}\,\beta^\mu.$$

(The difference betw. β^μ and θ^μ is that β^μ is a covector at one point $e \in G$, whereas θ^μ is a covector field, i.e., a 1-form. It is like the difference between $V_\mu \in \mathfrak{g}$ and $X_\mu \in \mathfrak{X}(G)$.) The forms θ^μ are <u>left-invariant 1-forms</u> on G. The set $\{\theta^\mu\}$ is dual to $\{X_\mu\}$ at each point $a \in G$, as we see by using the definitions,

$$\theta^\mu(X_\nu)\Big|_a = \theta^\mu\big|_a\left(X_\nu\big|_a\right) = \left(L^*_{a^{-1}}\beta^\mu\right)\left(L_{a*}V_\nu\right)$$

$$= \beta^\mu\left(L_{a^{-1}*}L_{a*}V_\nu\right) = \beta^\mu(V_\nu) = \delta^\mu_\nu.$$

Thus we have bases $\{X_\mu\}$ and $\{\theta^\mu\}$ ~~at each~~ of vectors and 1-forms at each point of G. These are generally non-coordinate bases (see HW). It is of interest to compute the components of $d\theta^\mu$ in this basis.

$$(d\theta^\mu)(X_\nu, X_\sigma) = \underbrace{X_\nu\,\theta^\mu(X_\sigma)}_{=X_\nu\,\delta^\mu_\sigma = 0} - \underbrace{X_\sigma\,\theta^\mu(X_\nu)}_{\text{also}=0} - \theta^\mu([X_\nu, X_\sigma])$$

$$= -\theta^\mu\left(C_{\nu\sigma}{}^\tau X_\tau\right) = -C_{\nu\sigma}{}^\mu.$$

So the structure constants (with a $-$ sign) are the components of $d\theta^\mu$ in the basis of LIVF's $\{X_\mu\}$. The 2-form $d\theta^\mu$ (in the abstract, for a fixed value of μ) is

$$\boxed{d\theta^\mu = -\frac{1}{2} C_{\nu\sigma}{}^\mu \; \theta^\nu \wedge \theta^\sigma.}$$ Maurer-Cartan structure equations.

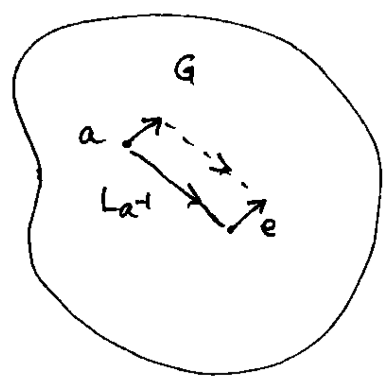
To put things in completely coordinate independent language, we write

$$\theta = V_\mu \otimes \theta^\mu.$$

θ is an example of a Lie-algebra valued 1-form. So far we have only seen real-valued 1-forms, ~~that is,~~ but Lie-alg. valued 1-forms are important in gauge theories (gauge potentials are such things).
θ is a map, (at a point $a \in G$)

$$\theta|_a : T_a G \to \mathfrak{g}.$$

It is easy to see abstractly what θ does: it uses left translation to map a vector in $T_a G$ to one in $T_e G = \mathfrak{g}$.



θ is called the Maurer-Cartan form. One might say that every Lie group carries on itself a gauge potential. The MC form can be written in fully coordinate-free notation if we define

$$d\theta = d(V_\mu \otimes \theta^\mu) = V_\mu \otimes d\theta^\mu,$$

logical since the V_μ are constant. This makes $d\theta$ a Lie-algebra-valued

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2-form. Also define

$$[\theta,\theta] = [V_\mu, V_\nu] \otimes \theta^\mu \wedge \theta^\nu,$$

another $\mathfrak{g}$-valued 2-form. Then the MC structure equations can be written,

$$\boxed{d\theta + \tfrac{1}{2}[\theta,\theta] = 0.}$$

Note, in QCD you get Lie-algebra valued 1-forms, this is the gauge potential, A^a_μ where $\mu = 0,..,3$ is a space-time index and $a = 1,\dots,8$ is an index ~~of~~ the basis in the $SU(3)$ Lie algebra, e.g., the index of the Gell-Mann matrices. Call these V_a. Then

$$V_a \, A^a_\mu \, dx^\mu$$

is a $\mathfrak{g}$-valued 1-form on space-time. (~~And~~ ~~...~~

(And, $F = \frac{1}{2} V_a \, F^a_{\mu\nu} \, dx^\mu \wedge dx^\nu$ is the Yang-Mills field tensor.)