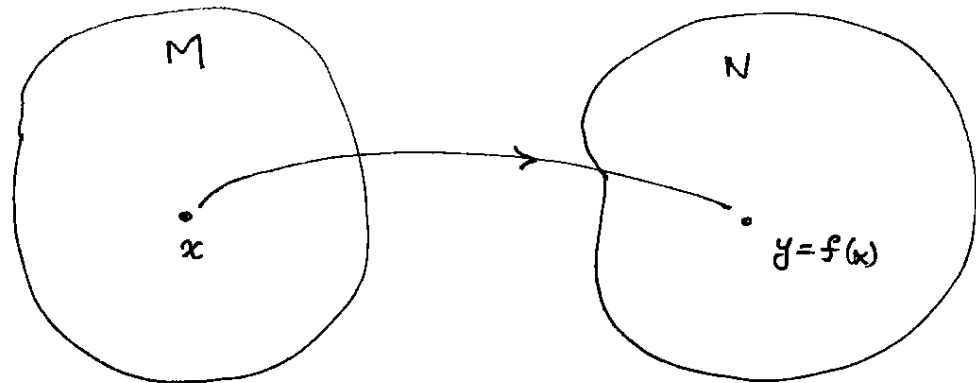


3/4/04

Behavior of differential forms under maps. Let $f: M \rightarrow N$ be a ^{smooth} map between two manifolds (not necessarily of same dimensionality). Let $x \in M$ and let $y = f(x) \in N$.



Let $\omega \in \Omega^r(N)$. Thus ω evaluated at $y = f(x)$ ~~is~~, denoted $\omega|_y = \omega|_{f(x)}$, is an operator acting on r tangent vectors $\in T_y N$. Now we define $f^* \omega \in \Omega^r(M)$ by showing its action on vectors $X_1, \dots, X_r \in T_x M$. Note that these are vectors at a point, not vector fields. Then the definition is

$$\boxed{f^* \omega|_x (X_1, \dots, X_r) = \omega|_{f(x)} (f_* X_1, \dots, f_* X_r).} \quad f_* = \text{tangent map.}$$

Doing this at each point $x \in M$ defines $f^* \omega$, the pull-back of ω under f . We previously defined the pull-back on scalars (0-forms), and on covectors (1-forms). This definition agrees with those in the cases $r=0$ or $r=1$.

3/4/04

①

Some comments. The exterior product of a set of 1-forms was defined by

$$(\alpha^1 \wedge \dots \wedge \alpha^r)(x_1, \dots, x_r) = \begin{vmatrix} \alpha^1(x_1) & \dots & \alpha^1(x_r) \\ \vdots & & \vdots \\ \alpha^r(x_1) & \dots & \alpha^r(x_r) \end{vmatrix} = \sum_{P \in \mathcal{S}_r} (-1)^P \alpha^1(x_{P_1}) \dots \alpha^r(x_{P_r}).$$

Note, regarding the tensor product,

$$(\alpha^1 \otimes \dots \otimes \alpha^r)(x_1, \dots, x_r) = \alpha^1(x_1) \alpha^2(x_2) \dots \alpha^r(x_r).$$

Therefore another definition of wedge product of r 1-forms is

$$\alpha^1 \wedge \dots \wedge \alpha^r = \sum_{P \in \mathcal{S}_r} (-1)^P \alpha^{P_1} \otimes \dots \otimes \alpha^{P_r}.$$

Example, $\alpha^1 \wedge \alpha^2 = \alpha^1 \otimes \alpha^2 - \alpha^2 \otimes \alpha^1.$

Now let $\omega \in \Omega^r(M)$. Then

$$\omega = \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_r}.$$

Actually, an expression like this holds for any type $(0, r)$ tensor.

But, since ω is antisymmetric, we also have

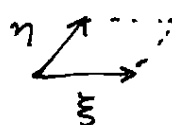
$$\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}.$$

This is a standard way of writing a diff. form in terms of its components.

Here the (implied) sum is, each $\mu_i = 1, \dots, m$ ($m = \dim M$). If you restrict to a definite order, you can drop the $\frac{1}{r!}$. Thus,

$$\omega = \sum_{\mu_1 < \mu_2 < \dots < \mu_r} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}.$$

Now the exterior derivative. Real motivation for this requires Stokes's theorem, for which later. For now just imagine integrating a 1-form α around a small parallelogram defined by two small vectors ξ, η :



$$\oint \alpha = \oint \alpha_\mu dx^\mu$$

Opposite sides obviously cancel to lowest order, so answer must involve derivatives of $\alpha_\mu(x)$. In fact, you find,

$$\oint \alpha = \frac{1}{2} \underbrace{(\alpha_{\nu, \mu} - \alpha_{\mu, \nu})}_{\text{antisymmetric in } \mu, \nu} (\xi^\mu \eta^\nu - \xi^\nu \eta^\mu).$$

interpreted as components of the 2-form $\beta = d\alpha$.

Idea of exterior derivative: map $d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$,

$$d\alpha = " \partial \wedge \alpha "$$

Define in coordinates. Let

$$\alpha = \frac{1}{r!} \alpha_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}.$$

Then

$$d\alpha = \frac{1}{r!} \alpha_{\mu_1 \dots \mu_r, \nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}.$$

Examples: Let $f \in \mathcal{F}(M) = \Omega^0(M)$.

$$df = f_{, \mu} dx^\mu. \quad \text{This is the covector } df, \text{ the}$$

"differential of a function" defined earlier.

3/4/04

Another ($r=1$) example of d . Let $A \in \Omega^1(M)$,

$$A = A_\mu dx^\mu$$

Then $dA = A_{\mu,\nu} dx^\nu \wedge dx^\mu$

$$= \frac{1}{2} \underbrace{(A_{\nu,\mu} - A_{\mu,\nu})}_{\substack{\text{components of } F = dA, \\ F \in \Omega^2(M)}} dx^\mu \wedge dx^\nu.$$

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

Example with $r=2$. Let $F \in \Omega^2(M)$,

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$dF = \frac{1}{2} F_{\mu\nu,\sigma} dx^\sigma \wedge dx^\mu \wedge dx^\nu$$

$$= \frac{1}{3!} \underbrace{(F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu})}_{\text{components of } dF} dx^\mu \wedge dx^\nu \wedge dx^\sigma$$

We recognize these examples from E+M ($A_\mu =$ vector potential, $F_{\mu\nu} =$ field tensor).

Properties of d .

(1) Distributive law on \wedge product. Let $\alpha \in \Omega^r(M)$, $\beta \in \Omega^s(M)$.

$$\text{Then } d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^r \alpha \wedge (d\beta).$$

(2) If $\alpha \in \Omega^r(M)$, then

$$d\alpha(x_1, \dots, x_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} x_i \alpha(x_1, \dots, \hat{x}_i, \dots, x_{r+1})$$

omit.

$$+ \sum_{i < j} (-1)^{i+j} \alpha([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{r+1}).$$

$$(3) \quad d^2 = 0.$$

Comments. Property (2) is equivalent to defn of d (since it gives action of dd on arb. set of vectors). Turns out properties (1)+(3) also imply defn of d . Prop. (1) follows easily from " $d = \partial \wedge$ " (you use a chain rule, but in order to bring the d in to act on β in the 2nd term, you must commute it through α , which introduces $(-1)^r$ factor.)

Proof of property (3):

$$\alpha = \frac{1}{r!} \alpha_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$$

$$d\alpha = \frac{1}{r!} \alpha_{\mu_1 \dots \mu_r, \nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \quad (\text{defn. of } d).$$

$$dd\alpha = \frac{1}{r!} \alpha_{\mu_1 \dots \mu_r, \nu \sigma} \underbrace{dx^\sigma \wedge dx^\nu}_{\text{symm antisymm}} \wedge \dots \wedge dx^{\mu_r} = 0.$$

Special cases of (2):

$$r=0, \quad f \in \mathcal{F}(M).$$

$$df(x) = Xf.$$

$$r=1, \quad \alpha \in \Omega^1(M)$$

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$$

$$r=2, \quad \beta \in \Omega^2(M)$$

$$d\beta(X, Y, Z) = X\beta(Y, Z) - Y\beta(X, Z) + Z\beta(X, Y) \\ - \beta([X, Y], Z) + \beta([X, Z], Y) - \beta([Y, Z], X).$$

(A) Another property of d . Let $f: M \rightarrow N$ be a map (not nec. a diffeo.). Let $\alpha \in \Omega^r(N)$, so $f^*\alpha \in \Omega^r(M)$. Then

$$\boxed{f^*(d\alpha) = d(f^*\alpha)} \quad d \text{ commutes with pull-backs.}$$

Easy to prove in components/coordinates.

Important terminology.

An r -form $\alpha \in \Omega^r(M)$ is closed if $d\alpha = 0$.

It is exact if $\exists \beta \in \Omega^{r-1}(M)$ such that $\alpha = d\beta$.

Now we consider the interior product. Let $X \in \mathfrak{X}(M)$, then the interior product is an operator $i_X: \Omega^r(M) \rightarrow \Omega^{r-1}(M)$, defined by ($\alpha \in \Omega^r(M)$):

$$(i_X \alpha)(Y_1, \dots, Y_{r-1}) = \alpha(X, Y_1, \dots, Y_{r-1}).$$

This is a purely algebraic operation (just insert X into 1st ~~to~~ slot of α), no differentiation required. Notice that i_X lowers the rank of α , while d raises it. Properties of i_X :

$$(1) \quad i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^r \alpha \wedge (i_X \beta), \quad \alpha \in \Omega^r(M)$$

$$(2) \quad i_X^2 = 0$$

$$(3) \quad \boxed{\mathcal{L}_X = i_X d + d i_X} \quad (\text{Acting on forms}).$$

Property (3) is the Cartan formula. The geometrical meaning of this formula must wait until we cover Stokes' theorem.

A proof is given in the book. The proof I prefer runs along

3/4/04 (6)

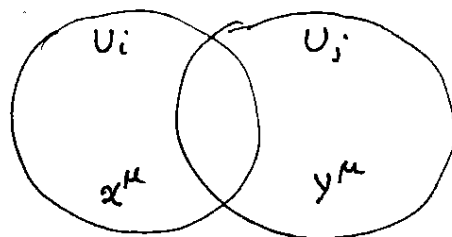
these lines:

- ① Show that the Cartan formula works for $r=0$ and $r=1$.
- ② Show that $i_x d + d i_x$ is a derivation (obeys Leibnitz) when acting on \wedge products.

These steps are straightforward. They prove the Cartan formula because an arbitrary form can be represented as a linear combination of \wedge products of 1-forms and 0-forms.

Now an introduction to the integration of differential forms. General idea is that r -forms get integrated over r -dimensional submanifolds of M . (Actually, the objects that they get integrated over are more general than submanifolds, they are chains. More about that later.) For now we concentrate on a special case, integrating m -forms on an m -dimensional manifold.

First, integration over a manifold is not meaningful unless the manifold is orientable. Consider 2 charts that overlap. ($m = \dim M$).



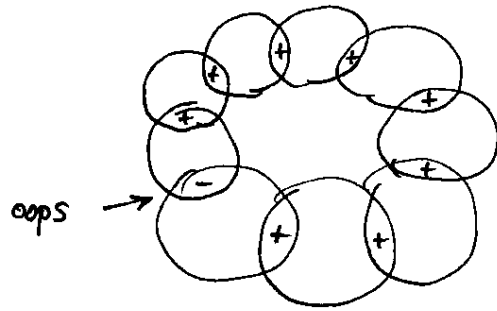
$$\varphi_i : U_i \rightarrow \mathbb{R}^m$$

In the overlap region, the Jacobian $\frac{\partial x^\mu}{\partial y^\nu}$ is nonsingular, so $\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right)$ is either > 0 or < 0 . If it is > 0 , then we say the two charts have the same orientation. Is it possible to cover M with charts that have the same orientation? Depends on M . Those manifolds that can be covered with charts

that have the same orientation are said to be orientable.

3/4/04 7

For some manifolds, however, this cannot be done.



Example of Möbius strip,
 \mathbb{RP}^2

There is a relation between orientability and m -forms on M . Let $\phi \in \Omega^m(M)$.
Then

$$\phi = \frac{1}{m!} \phi_{\mu_1 \dots \mu_m}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m}$$

$$= \underbrace{\phi_{12 \dots m}(x)} dx^1 \wedge \dots \wedge dx^m$$

↳ Call this $\rho(x)$, it is the one component of ϕ .
w.r.t. chart x^μ . indep

change charts, $x^\mu \rightarrow y^\mu$. Then

$$dx^\mu = \frac{\partial x^\mu}{\partial y^\nu} dy^\nu.$$

$$dx^1 \wedge \dots \wedge dx^m = \det\left(\frac{\partial x}{\partial y}\right) dy^1 \wedge \dots \wedge dy^m, \text{ because of antisymmetry.}$$

So, ~~in~~ under the change of coordinates $x^\mu \rightarrow y^\mu$, we find $\rho \rightarrow \rho \det\left(\frac{\partial x}{\partial y}\right)$. We say that ρ transforms

as a pseudo-scalar. The value of ρ is not preserved under a change of coordinates, but if x^μ and y^μ ~~are~~ have the same orientation, then the sign of ρ is conserved.

Related to a thm: \otimes An m -form ϕ exists on M that is nonzero everywhere iff M is orientable. This is easily proved using partitions of unity, discussed in book. Note that ϕ vanishes at a point $x \in M$ iff $\rho(x) = 0$.

If M is orientable, then we can construct atlases in which all the charts have the same orientation. Call these "oriented atlases". If we have two atlases, their charts are all either oriented the same or oriented oppositely (if they overlap). Thus the space of oriented atlases consists of two equivalence classes. We may call one of these "positively oriented" and the other "negatively oriented", but this is just a convention, not an absolute designation.

In order to specify an orientation, we choose an oriented atlas. Actually it suffices to choose a single chart, or even just a basis $\{e_\mu\}$ in a single tangent space at a single point, since if M is orientable this will ~~lock~~^{fix} the orientations of all other charts.

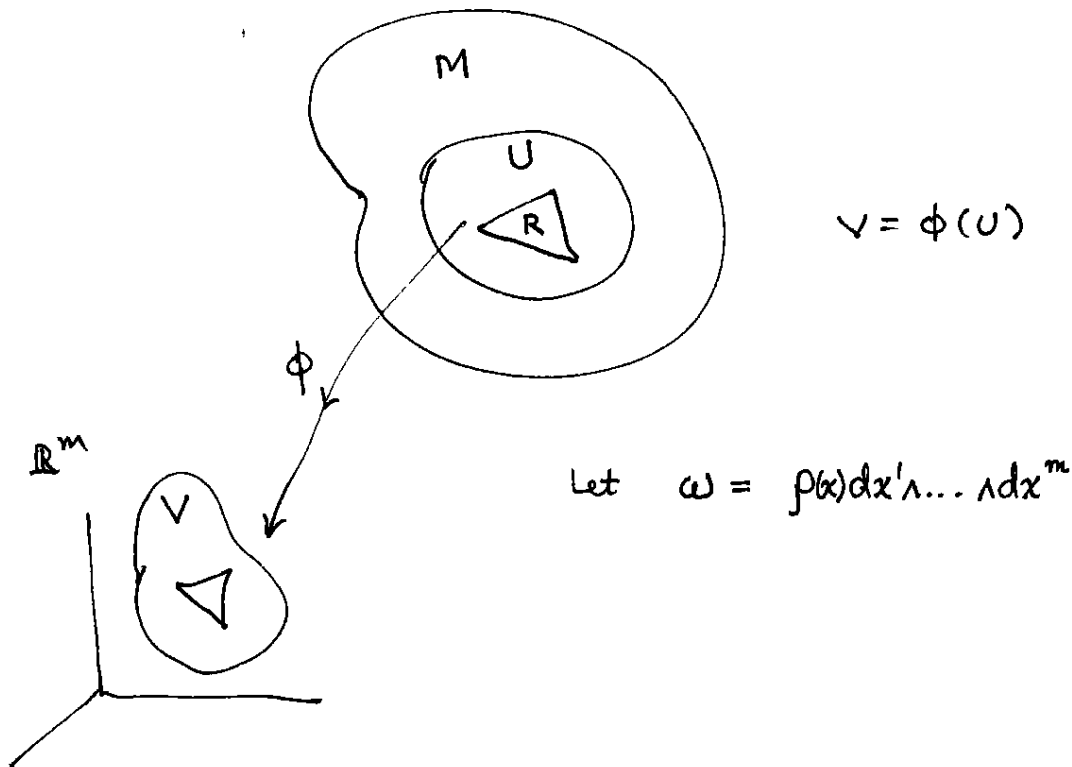
Let $\omega \in \Omega^m(M)$, and suppose we have an oriented atlas chosen.

~~Then we triangulate R so that each~~ Then to integrate ω over a region contained in one chart x^μ we define

$$\int_R \omega = \int dx^1 \dots dx^m$$

Let $\phi: U \rightarrow \otimes V \subset \mathbb{R}^m$ be a chart containing a region R ($R \subset U$) over which we wish to integrate ω .

3/4/04



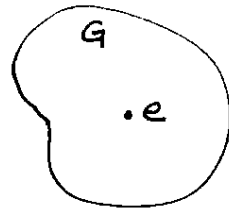
Then we define

$$\int_R \omega = \int_{\phi(R)} \rho(x) dx^1 \dots dx^m.$$

The final integral is a normal Riemann integral (in particular, the answer does not depend on the ordering of the dx 's.) By breaking M up into regions such that each region lies in one chart, we can ~~can~~ add the integrals up to get $\int_M \omega$. The answer is independent of the oriented atlas we choose, apart from sign.

(10)
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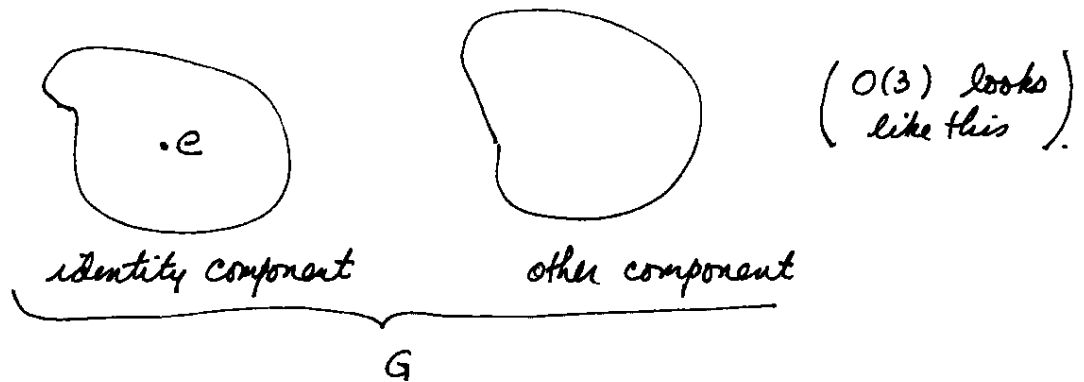
Now we turn to the differential geometry of Lie groups. A Lie group is a group that is also a differentiable manifold, in which the group operations of multiplication and forming the inverse are smooth. Denote the identity by e . We may indicate it schematically by



← as a manifold, G has a dimensionality, the dimension of G .

Think of $SO(3) = \mathbb{RP}^3$ and $SU(2) = S^3$.

but we must remember that some groups consist of more than one disconnected component, for example $O(3)$ consists of two disconnected components, these orthog. matrices with $\det R = \pm 1$, and the Lorentz group $SO(3,1)$ consists of 4 disconnected components. One of the components must contain the identity (the identity component).



A discrete group can be thought of as a Lie group of dimension 0 (with each group element constituting a single component).

→ orthogonal, unitary, symplectic, etc.
Matrix groups are particularly important examples in physics.

These can be seen as submanifolds of "matrix space", which is \mathbb{R}^{n^2} or $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ for real or complex matrices.

We will start by developing the differential geometry of a Lie group intrinsically, i.e., without reference to an embedding in "matrix space" or (any other embedding).

But first some material on group actions. All of this was discussed in various homeworks.

Let $M =$ a space and G a group. An action of G on M is a map: $g \mapsto \Phi_g$, where $\Phi_g: M \rightarrow M$, such that

$$\Phi_{g_1} \Phi_{g_2} = \Phi_{g_1 g_2}. \quad (\text{implies } \Phi_e = \text{id}_M).$$

(also called a left action). Since $\Phi_g \Phi_{g^{-1}} = \Phi_e = \text{id}_M$, it means.

Φ_g is a bijection, in fact, the group action can be seen as a homomorphism, $G \rightarrow \{\text{Bijections of } M \rightarrow M\}$. When M is a manifold, we will be interested in smooth bijections, i.e., diffeomorphisms. If M is a vector space V and Φ_g are linear maps $\Phi_g: V \rightarrow V$, then the action is called a representation.

If $x_0 \in M$, then the set $\{\Phi_g x_0 \mid g \in G\}$ is the orbit of x_0 under the action. Don't confuse this with orbits in classical mechanics (note, however, that an orbit in phase space in classical mechanics is ^{also} an orbit in the ~~usual~~ group action mathematical sense, where the action is $t \mapsto \Phi_t =$ the advance map of some vector field, and $G = \mathbb{R}$). The orbits of \bullet some action of G on M are disjoint subsets of M (M is divided or "foliated" into disjoint subsets).

Given $x_0 \in M$, the set of group elements that leave x_0 invariant, $I_{x_0} = \{g \in G \mid \Phi_g x_0 = x_0\}$, is called