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Notation Summary.

$$T_p M = \{ \text{tangent vector } X \text{ at } p \in M \}$$

$$T_p^* M = \{ \text{cotangent vectors } \alpha \text{ at } p \in M \} \quad \alpha: T_p M \rightarrow \mathbb{R}.$$

$$TM = \bigcup_{p \in M} T_p M = \text{tangent bundle}, \quad \pi: TM \rightarrow M, \quad \pi(T_p M) = p$$

$$T^*M = \bigcup_{p \in M} T_p^* M = \text{cotangent bundle}, \quad \pi: T^*M \rightarrow M, \quad \pi(T_p^* M) = p.$$

$$\text{Vector field } X: M \rightarrow TM, \quad \pi(X(p)) = p.$$

$$\text{Covector field or 1-form } \alpha: M \rightarrow T^*M, \quad \pi(\alpha(p)) = p.$$

$$\mathcal{F}(M) = \{ \text{smooth scalar fields on } M \}$$

$$\mathcal{X}(M) = \{ \text{smooth vector fields on } M \}$$

$$\mathcal{X}^*(M) = \Omega^1(M) = \{ \text{smooth covector fields on } M \}.$$

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Recall, a tensor of type  $(r,s)$  at a point  $p \in M$  is a multilinear map,

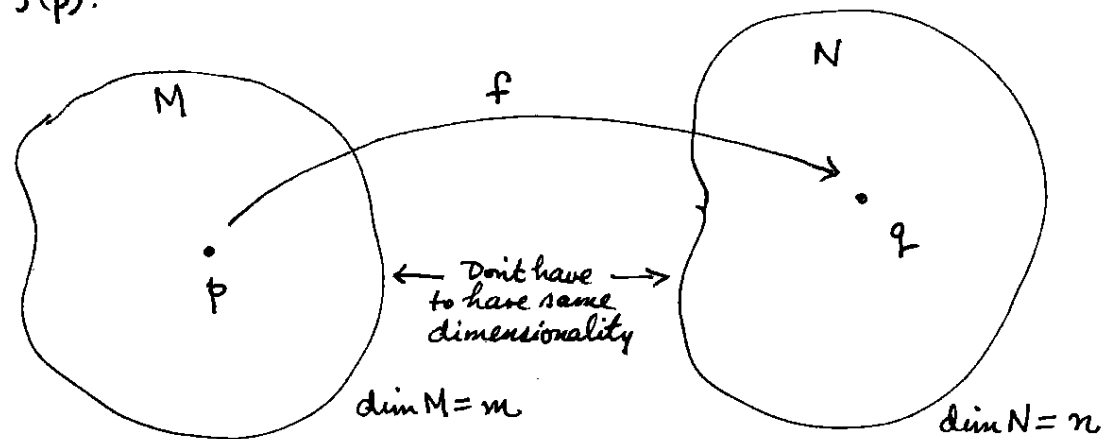
$$T: \underbrace{T_p^*M \times \dots \times T_p^*M}_r \text{ copies} \times \underbrace{T_pM \times \dots \times T_pM}_s \text{ copies} \rightarrow \mathbb{R}.$$

The components of  $T$  are given by (w.r.t. a chart)

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} = T(dx^{i_1}, \dots, dx^{i_r}; \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}}).$$

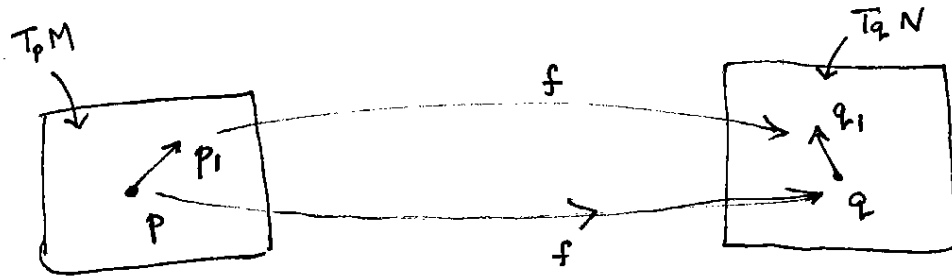
A tensor field on  $M$  is an assignment of a tensor at each point  $p \in M$ . The components of a tensor field are functions of position.

Now we consider the behavior of fields under maps. Let  $f: M \rightarrow N$  be a map between manifolds, let  $p \in M$  and  $q \in N$  be points such that  $q = f(p)$ .

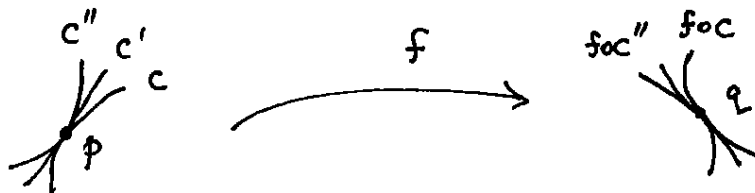


Question: Given  $X \in T_pM$ , is there any way to associate it with a  $Y \in T_qN$ ? Yes, use the small displacement interpretation of a tangent vector ( $p_1$  close to  $p$ ), and define  $q_1 = f(p_1)$ . (You map both the base and the tip of the small arrow under  $f$ ,

to get a new small arrow on  $N$ . Both are understood to be displacements taking place in some elapsed parameter  $\Delta t$ .)



Alternatively, in the equivalence class of curves interpretation, just map the curves themselves:



Thus we can say, if  $X = [c]$ , then  $Y = [f o c]$ . This defines

a linear map

$$f_* : T_p M \rightarrow T_q N$$

where  $f_*$  is called the tangent map, derivative map, or push-forward. Note that  $f_*$  can be defined succinctly by

$$f_* [c] = [f o c].$$

If we impose coordinates (charts) on  $M$  and  $N$  (containing  $p$  and  $q$ ), we can write  $f_*$  in coordinates. Let  $x^i$  be coordinates on  $M$  and  $y^i$  those on  $N$ . Let

$$X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i} \Big|_p$$

$$f_* X = Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial y^i} \Big|_q$$

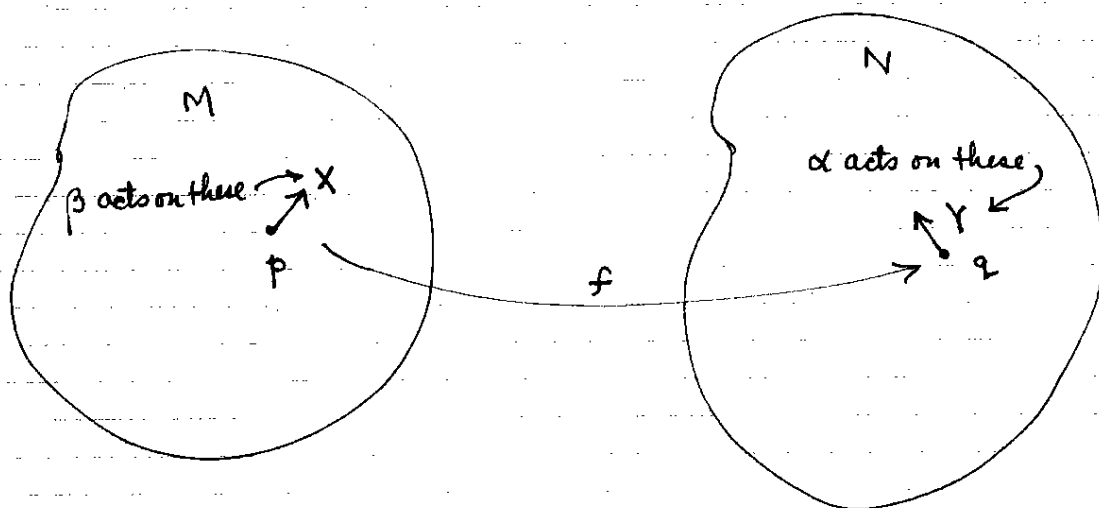
Then

$$Y^i = \sum_j \frac{\partial y^i}{\partial x^j} X^j,$$

this is just the chain rule, 2/26/04  
 ← if you think  $X^i = \frac{dx^i}{dt}$ ,  $Y^i = \frac{dy^i}{dt}$ .

where  $\partial y^i / \partial x^j$  is the coordinate representation of the derivatives of  $f$ .

This is how we map vectors. For covectors, it works the other way, i.e., given a covector  $\alpha \in T_q^* N$ , we can associate it with another covector  $\beta \in T_p^* M$ .



That is, we define a map  $f^*: T_q^* N \rightarrow T_p^* M$  by demanding  $\beta(X) = \alpha(Y)$  when  $Y = f_* X$ . That is,  $\beta = f^* \alpha$  is defined by

$$(f^* \alpha)(X) = \alpha(f_* X), \quad \forall X \in T_p M.$$

The covector  $f^* \alpha \in T_p^* M$  is said to be the pull-back of  $\alpha \in T_q^* N$ , because it works in the opposite direction to  $f$ .

This is for a covector at a point (two of them,  $\alpha$  and  $f^* \alpha$ ). If now we let  $\alpha$  be a covector field (same symbol, new meaning),  $\alpha \in \mathcal{X}^*(N)$ , then  $f^* \alpha \in \mathcal{X}^*(M)$ , is given by

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$$(f^*\alpha)_p(X) = \alpha_{f(p)}(f_*X), \quad \forall p \in M$$

where the subscript indicates the point at which the field is evaluated.

Thus 1-forms on  $N$  get pulled back into 1-forms on  $M$ .

A simpler example of the pull-back is for scalar fields.

Let  $\phi \in \mathcal{F}(N)$ ,  $\phi: N \rightarrow \mathbb{R}$  be a scalar field. Then we define the pull-back  $\psi = f^*\phi \in \mathcal{F}(M)$  by

$$\psi(p) = (f^*\phi)(p) = \phi(f(p)) = (\phi \circ f)(p),$$

that is,  $f^*\phi = \phi \circ f$ .

~~This maps~~ Return to the pull-back of covectors, and put it into coordinate language. We let  $x^i$  and  $y^i$  be coordinates on  $M$  and  $N$  as above, we let  $\beta = f^*\alpha$ , we write

$$\alpha = \sum_{i=1}^n \alpha_i dy^i|_q$$

$$\beta = \sum_{i=1}^m \beta_i dx^i|_p$$

Then

$$\beta_i = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i} \alpha_j.$$

Notice that  $f^{-1}$  need not be defined in order to define  $f_*$  and  $f^*$ . In particular,  $M$  and  $N$  need not have the same dimensionality.

But if  $f^{-1}$  does exist (then  $\dim M = \dim N$ ), then we can

"push-forward" covectors from  $M$  to  $N$  (by  $f^{-1*}$ ), and

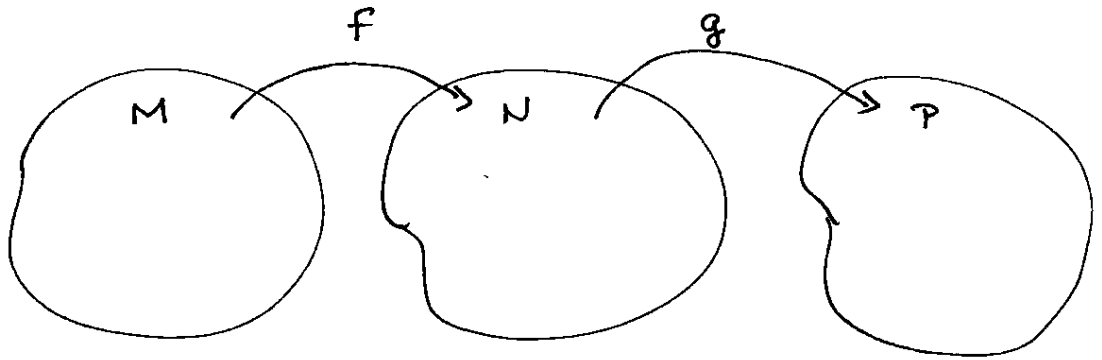
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Behavior of  $f^*$ ,  $f_*$  under compositions. Let  $f: M \rightarrow N$ ,

$g: N \rightarrow P$



Then

$$g \circ f : M \rightarrow P,$$

and

$$(g \circ f)_* = g_* \circ f_*.$$

This is fairly obvious, you just map the small displacement vector (in M) under a succession of 2 linear maps, first by  $f_*$ , then  $g_*$ , to get  $(g \circ f)_*$ .

As for pull-backs, the rule is

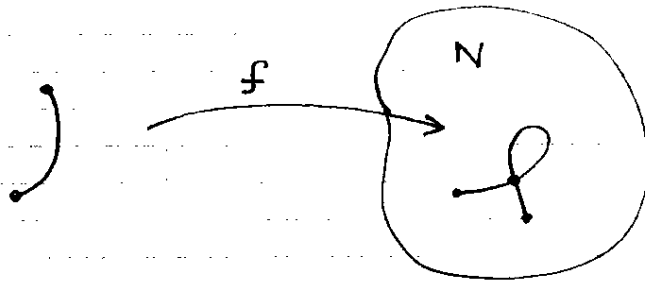
$$(g \circ f)^* = f^* \circ g^* \quad (\text{in reverse order}).$$

"pull-back" vectors from  $N$  to  $M$  (by  $f_*^{-1}$ ).

Now consider the mapping ~~one~~ of one manifold of a certain dimensionality into one of higher dimensionality,  $f: M \rightarrow N$ ,  $\dim M \leq \dim N$ . Then  $f$  is called an immersion if  $f_*$  is of maximal rank,

$$\text{rank } f_*: T_p M \rightarrow T_{f(p)} N = \dim M.$$

This means that each little piece of  $M$ , which looks like  $\mathbb{R}^m$ , gets mapped into a subset of  $N$  that also looks like  $\mathbb{R}^m$ . This means that  $f_*$  is injective (the image of  $M$  under  $f$  is locally  $\overset{\leq \dim M}{m}$ -dimensional). However, an immersion does not preclude self-intersections:



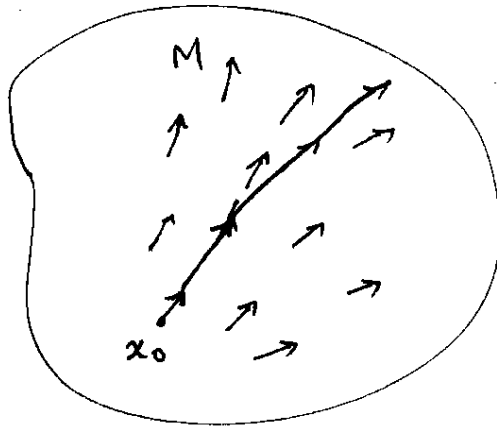
To exclude self-intersections, we can demand that  $f$  itself be an injection. ~~This means~~ Then  $f$  is called an embedding (because  $\text{im } f$  "looks like"  $M$ ).

Now we consider ordinary differential equations (ODE's) and flows.

Begin with an intuitive picture of a vector field, as a small displacement (each understood to be taking place in some elapsed parameter  $\Delta t$ ), attached to each point of  $M$ :

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Picture of  
 $X \in \mathfrak{X}(M)$



Idea: Each point gets up and moves a small amount in time  $\Delta t$ .

If you just follow these arrows, starting with some initial point  $x_0$ , you trace out a curve called the integral curve of  $X$ . By following in integral curve for time  $t$ , starting at  $x_0$ , you get a final point  $x$  described by a function

↙ at least sometimes it has this domain

$$\Phi: M \times \mathbb{R} \rightarrow M,$$

$$x = \Phi(x_0, t),$$

where  $\Phi$  is called ~~the~~ <sup>the</sup> advance map.

To make this more precise, express the vector field  $X$  in some chart:

$$X = \sum_i X^i(x) \frac{\partial}{\partial x^i}$$

$x$ = a point
$x^i$ = its coordinates
$X$ = a vector
$X^i$ = its components.

This is an operator which when acting on scalars  $f$  gives a number interpreted as  $df/dt$ . In particular, letting it act on the coordinates themselves gives a set of ODEs:

$$\frac{dx^i}{dt} = X^i(x).$$

Thus, a vector field on a manifold is a generalization of a system of ODE's on  $\mathbb{R}^n$ . Standard theorems on ODE's say that the system above has a unique solution  $x^i(t)$  satisfying  $x^i(0) = x_0^i$  for  $t$  in some interval



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$[0, \infty]$ , if the functions  $X^i(x)$  are smooth. This is the (important)

uniqueness theorem for ODE's (really, existence and uniqueness).

However, even if the vector field  $X^i(x)$  is smooth, the solution may not exist for all  $t$  (for example, it may run off to infinity in finite  $t$ ). (For an example of this, consider  $\dot{x} = x^2$ ,  $x_0 = 1$  ( $x \in \mathbb{R}$ ), for which  $x \rightarrow \infty$  as  $t \rightarrow 1$ .)

For simplicity, we will assume that this does not happen, i.e., that solutions  $x^i(t)$  exist for all time, for any  $x_0^i$ . Then we can speak of the "general solution functions"

$\Phi^i(t, x_0)$  that give  $x^i(t)$ , assuming  $x^i(0) = x_0^i$ . These ~~solution~~ functions satisfy, [we write  $x^i(t) = \Phi^i(t, x_0)$ , indicating explicitly the dependence on i.c.'s]

$$1) \quad \Phi^i(0, x_0) = x_0^i$$

$$2) \quad \frac{\partial \Phi^i}{\partial t}(t, x_0) = X^i(\Phi(t, x_0)).$$

These are just the initial conditions and ODE's expressed in terms of  $\Phi^i$ . [We would normally write them,

$$1) \quad x^i(0) = x_0^i$$

$$2) \quad \frac{dx^i}{dt} = X^i(x(t)) \quad ].$$

All of the above is in one chart. By mapping a solution  $x^i(t)$  in the given chart back onto  $M$ , we get a segment of an integral curve. Just before we run off one chart we can switch to another, thereby continuing the integral curve.

The result is that we define a map  $\Phi: \mathbb{R} \times M \rightarrow M$

or maps  $\Phi_t: M \rightarrow M$  (a different notation), such that

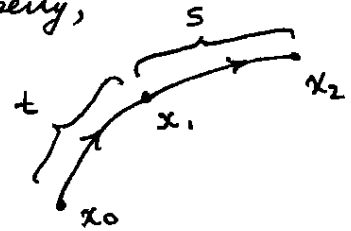
$$x = \Phi(t, x_0) = \Phi_t(x_0)$$

is the point on ~~an~~ <sup>the</sup> integral curve starting at  $x_0$  at  $t=0$ , reached after time  $t$ .

The advance map satisfies an important property,

$$\Phi_s \Phi_t = \Phi_{s+t}$$

$$\text{or } \Phi(s, \Phi(t, x_0)) = \Phi(s+t, x_0).$$



(the composition property). This is intuitive: if you start at  $x_0$ , follow ~~to~~ the integral curve for elapsed time  $t$ , reaching  $x_1$ , then treat  $x_1$  as initial conditions and follow the integral curve for elapsed time  $s$ , you must get the same thing as starting at  $x_0$  and following the integral curve for time  $s+t$ .

We will prove this working in a single chart, ignoring the complications that result when we must switch charts. We want to show that

$$\Phi^i(s, \Phi(t, x_0)) = \Phi^i(s+t, x_0).$$

$$\text{Let } x_1^i = \Phi^i(t, x_0), \quad \xi^i(s) = \Phi^i(s, x_1), \quad \eta^i(s) = \Phi^i(s+t, x_0).$$

We need to show that  $\xi^i(s) = \eta^i(s)$ . First, at  $s=0$ , we have

$$\xi^i(0) = \Phi^i(0, x_1) = x_1^i$$

$$\eta^i(0) = \Phi^i(t, x_0) = x_1^i.$$

Next, we have

$$\frac{d\xi^i}{ds} = \frac{\partial \Phi^i}{\partial s}(s, x_1) = \mathbb{X}^i(\Phi(s, x_1)) = \mathbb{X}^i(\xi(s)),$$

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and

$$\begin{aligned} \frac{d\eta^i}{ds} &= \frac{\partial \Phi^i}{\partial s}(s+t, x_0) = \frac{\partial \Phi^i}{\partial (s+t)}(s+t, x_0) \\ &= \mathbb{X}^i(\Phi(s+t, x_0)) = \mathbb{X}^i(\eta(s)). \end{aligned}$$

Thus, both  $\xi^i(s)$  and  $\eta^i(s)$  satisfy the same ODE's and the same initial conditions, so by the uniqueness theorem they must be equal. QED.

By the composition property,  $\Phi_t \circ \Phi_t = \text{id}_M$ , so  $\Phi_t$  is a diffeomorphism:  $M \rightarrow M$ . In fact, the set

$$\{\Phi_t \mid t \in \mathbb{R}\}$$

constitutes a one-parameter group of diffeomorphisms of  $M$  onto itself. It is an action of the group  $\mathbb{R}$  (meaning  $t$ ) on  $M$ . This group is sometimes called the flow.

Now about the exponential notation for the flow. This is a way of connecting  $\Phi_t$  with the vector field  $\mathbb{X}$ . The notation used in many books is

$$\boxed{\Phi_t = e^{t\mathbb{X}}}$$

Taken literally ~~as written~~, this has no meaning (we must assign a meaning to it). First note that a vector field  $\mathbb{X}$  is a mapping:  $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$ , namely,  $f \mapsto \sum_i X^i \frac{\partial f}{\partial x^i}$ . Therefore  $\mathbb{X}^2 = \mathbb{X} \cdot \mathbb{X} = \mathbb{X} \circ \mathbb{X}$  has a meaning as a map:  $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$ . The higher powers of a vector field are not vector fields (they are not 1st order partial differential operators, they are higher order diff. ops.), but they are perfectly good maps:  $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$ . [Notice that a vector at a point is a map:  $\mathcal{F}(M) \rightarrow \mathbb{R}$ , so a power of it has no meaning.]

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So, an exponential series like

$$e^{tX} = 1 + tX + \frac{t^2}{2!} X^2 + \dots$$

has meaning as a map:  $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$ , at least if we ignore convergence questions (which we will). The 1 above means  $\text{id}_M$ .

On the other hand,  $\Phi_t$  is a map:  $M \rightarrow M$ , not:  $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$ . This is why  $\Phi_t = e^{tX}$  has no meaning as it stands. But let us apply the exponential series to a function  $f$  and see what we get. We evaluate the function at  $x_0$ , which is an initial condition. ~~We write  $e^{tX}$  instead of  $\Phi_t$ .~~

$$\begin{aligned} e^{tX} f &= f + tXf + \frac{t^2}{2} X^2 f + \dots \\ &= f + t \sum_i X^i \frac{\partial}{\partial x^i} f + \frac{t^2}{2} \sum_i X^i \frac{\partial}{\partial x^i} \sum_j X^j \frac{\partial}{\partial x^j} f + \dots \\ &= f + t \text{ " } \frac{df}{dt} \text{ " } + \frac{t^2}{2} \text{ " } \frac{d^2 f}{dt^2} \text{ " } + \dots \end{aligned}$$

where we put the  $t$ -derivatives in quotes because what is actually meant is

$$\text{" } \frac{df}{dt} \text{ " } = \frac{d}{dt}(f \circ c)$$

where  $c$  is the integral curve passing through  $x$  at  $t=0$ . Thus, if the series converges, it does so to "  $f(t)$  ", which means  $f(x(t))$ , where

$x(t)$  is the integral curve,  $x(t) = \Phi_t(x_0)$ . So,

$\uparrow$  means same as  $c(t)$

$$(e^{tX} f)(x_0) = f(\Phi_t(x_0)) = (\Phi_t^* f)(x_0).$$

This is true for all  $x_0$ , so we have

$$e^{tX} f = \Phi_t^* f.$$

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And this is true for all  $f$ , so we have

$$\Phi_t^* = e^{tX} \quad (A)$$

The usual formula in books is meaningful if we put a  $*$  on  $\Phi_t$  (turning it into a pull-back), and interpret both sides as maps:  $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$ .

On the other hand, we can regard  $e^{tX}$  as a formal notation for  $\Phi_t$ , without trying to interpret it as a power series. This notation has the virtue of making some of the properties of the advance map obvious:

$$\Phi_s \Phi_t = e^{sX} e^{tX} = e^{(s+t)X} = \Phi_{s+t}$$

$$\text{and } \Phi_0 = e^{0X} = 1 = \text{id}_M.$$

Restoring the star  $*$ , we can differentiate  $\Phi_t^* = e^{tX}$  formally, to get

$$\frac{d}{dt} \Phi_t^* = X e^{tX} = e^{tX} X, \quad \cancel{\frac{d}{dt} \Phi_t^* = \Phi_t^* X}$$

which implies

$$\frac{d}{dt} \Phi_t^* = X \Phi_t^* = \Phi_t^* X, \quad (B)$$

an equation that is perfectly meaningful as operators:  $\mathcal{F}(M) \rightarrow \mathcal{F}(M)$ . We have not proved it (because of questions of convergence of series), but in fact this result is true, and can be proved by other means.

Altogether, we have derived relationships between a vector field  $X$  and the 1-parameter group of diffeomorphisms  $\{\Phi_t\}$  that it generates, in an intrinsic notation (not tied to a coordinate system). These are Eqs. (A) and (B) above.