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Summary:

$\vec{\omega}(t) = g \frac{e}{2mc} \vec{B}(t)$ $\qquad$ $\vec{S}(t) = \langle \chi(t) | \frac{\hbar}{2}\vec{\sigma} | \chi(t) \rangle$

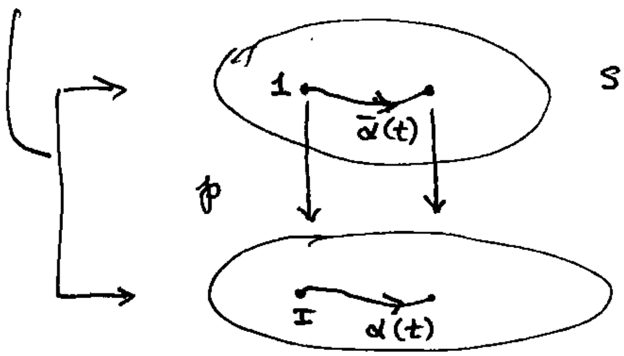
$$\left.\begin{aligned} i\hbar \frac{\partial \chi}{\partial t} &= \vec{\omega}(t)\cdot\left(\frac{\hbar}{2}\vec{\sigma}\right)\chi \\ \frac{d\vec{S}}{dt} &= \vec{\omega}(t) \times \vec{S} \end{aligned}\right\} \qquad \left.\begin{aligned} \chi(t) &= U(t)\,\chi_0 \\ \vec{S}(t) &= R(t)\,\vec{S}_0 \end{aligned}\right\} \text{ solns.} \quad \begin{aligned} U(t) &\in SU(2) \\ R(t) &\in SO(3) \end{aligned}$$

View $U(t)$, $R(t)$ as paths in $SU(2)$, $SO(3)$, write $\alpha(t) = R(t)$, $\bar{\alpha}(t) = U(t)$,

not realistic pictures of these manifolds.

$\alpha : [0,T] \longrightarrow SO(3)$ $\qquad$ $\alpha(0) = I$

$\bar{\alpha} : [0,T] \longrightarrow SU(2)$ $\qquad$ $\bar{\alpha}(0) = 1$



$SU(2) = S^3$

$SO(3) = \mathbb{R}P^3$

$$\left.\begin{aligned} &p = \text{projection} \\ &p : SU(2) \to SO(3) \\ &R = p(U) \\ &R_{ij} = \tfrac{1}{2}\operatorname{tr}\left(U^{+}\sigma_i U \sigma_j\right) \end{aligned}\right\}.$$

$p(U) = p(-U)$.

α is the projection of $\bar{\alpha}$, $\qquad$ $\alpha = p \circ \bar{\alpha}$, $\quad \alpha(t) = p(\bar{\alpha}(t))$.

$\bar{\alpha}$ is the <u>lift</u> of α.

$\pi_1(SU(2)) = \{e\}$

$\pi_1(SO(3)) = \mathbb{Z}_2$

There are two purposes of this spin-in-$\vec{B}$-field example. (1) To illustrate topological facts, $SO(3) = \mathbb{R}P^3$, $SU(2) = S^3$, and the geometrical interpretation of the 2-to-1 projection p. (2) To give an example of a covering space, with projection p, and lift of curves.

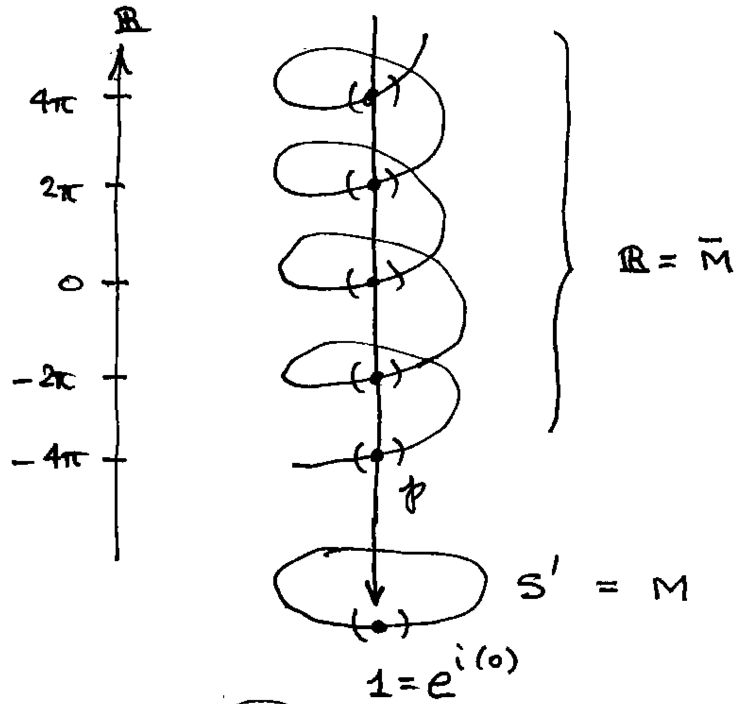
Concering (1), the geometrical meaning of $p: SU(2) \to SO(3)$ or $p: S^3 \to \mathbb{R}P^3$ is just the identification of antipodal points on S^3 to produce an element of $\mathbb{R}P^3$. You can see this because both U and $-U$ project onto the same R, and $U, -U$ are antipodal points in $\mathbb{R}^4$, using the Cayley-Klein parameters.

Concerning (2), terminology is that $SU(2)$ is the <u>covering</u> <u>space</u> of $SO(3)$, in fact, it is an example of a <u>universal</u> <u>covering</u> <u>space</u>, p is the projection, and $\bar{\alpha}$ is the <u>lift</u> of α (or $U(t)$ is the lift of $R(t)$). Definitions given later, for now just remark that a <u>covering</u> <u>space</u> is in a sense an "unrolling" of a space that is not simply connected to make a larger space. The unrolling can be either partial or complete; if it is complete, then the unrolled or covering space is <u>simply</u> <u>connected</u>, and it is called the "universal" covering space. Note that $S^3 = SU(2)$ is simply connected; it is the U.C.S. of $SO(3) = \mathbb{R}P^3$.

The simplest example of a universal covering space is that obtained by unrolling a circle into a line. Let $M = S^1$, and $\bar{M}$ (the covering space) be $\mathbb{R}$. Identify S^1 with the unit circle in the complex plane, and define $p: \mathbb{R} \to S^1$ (the projection) by

$$p : \mathbb{R} \to S^1 : x \mapsto e^{ix}.$$

The covering space $\mathbb{R}$ can be seen as a "helix" over S^1. Note that the preimage of $1 = e^{i(0)}$ in S^1 is the set of points $\{2\pi n \mid n \in \mathbb{Z}\}$ in $\mathbb{R}$:



The points in $p^{-1}(x)$ can be regarded as the "branches" of the "multi-valued inverse function" p^{-1}.

If you choose an open interval around $1 \in S^1$ that is small enough that it is simply connected, then it has an infinite set of preimages in $\mathbb{R}$, each of which is homeomorphic to the original interval in S^1. This means that $\mathbb{R}$ "looks like S^1" locally.

Now we will use this picture to prove that $\pi_1(S^1) = \mathbb{Z}$. First we explain the meaning of the lift of a curve. Let $\alpha : [0,1] \to S^1$ be a continuous path in $M = S^1$, such that $\alpha(0) = 1 = e^{i(0)}$. Let $\bar{x}_0$ be a point in $\bar{M} = \mathbb{R}$ such that $p(x_0) = 1$. This means $\bar{x}_0 = 2\pi n$ for some $n \in \mathbb{Z}$ (it is a choice of a preimage of $\alpha(0)$).

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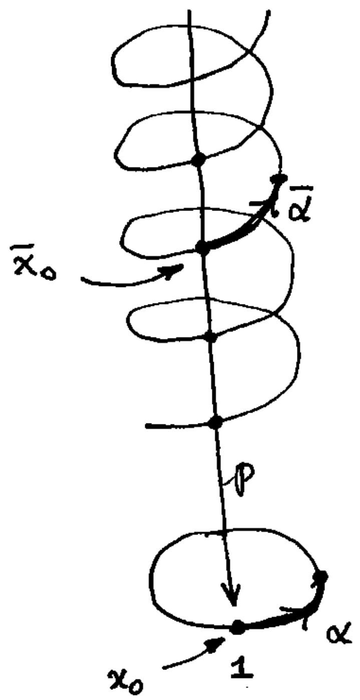

$\bar{\alpha}$ = <u>lift</u> of α, satisfying i.e. $\bar{\alpha}(0) = \bar{x}_0$ = selected point in preimage of $x_0 = 1$

Then we take it as obvious that there exists a <u>unique</u> continuous path $\bar{\alpha} : [0,1] \to \bar{M} = \mathbb{R}$ satisfying the initial conditions $\bar{\alpha}(0) = \bar{x}_0$ such that $\alpha = p \circ \bar{\alpha}$, i.e. $\alpha(t) = p(\bar{\alpha}(t))$. This is because as you move along $\alpha(t)$, there is a discrete set of choices for a preimage $\bar{\alpha}(t)$ that can be made, but since each of these is separated from one another, and since $\bar{\alpha}$ must be continuous, then there can be only one choice for a preimage, once an initial condition has been chosen.

$(I = [0,1])$

Now let $\alpha : I \to S^1$ be a <u>loop</u> based at $1 = e^{i(0)}$, so that $\alpha(0) = \alpha(1) = 1$, let $\bar{x}_0 = 0 = \bar{\alpha}(0)$, and let $\bar{\alpha} : I \to \mathbb{R}$ be the <u>lift</u> of $\alpha : I \to S^1$. Then $p(\bar{\alpha}(1)) = 1 = e^{i(0)}$, so $\bar{x}_1 = \bar{\alpha}(1) = 2\pi n$ for some $n \in \mathbb{Z}$. We define n as the <u>winding number</u> of the loop α. We wish to show that the winding number uniquely characterized the homotopy class to which α belongs.

First, we show that if $\alpha \sim \alpha'$, then $n = n'$ (α homotopic to α').

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This follows from continuity. As we deform loop α into loop α', the endpoint $\bar{\alpha}(1)$ of the lifted curve cannot change, because any change would be discontinuous. Therefore $\bar{\alpha}'(1) = \alpha(1) = 2\pi n$ (the same n).

Next, we show that if we have two loops $\alpha, \alpha' : I \to S^1$ with the same winding number, $n = n'$ then they are homotopic. Proof: the lifted paths $\bar{\alpha}, \bar{\alpha}'$ both start at $\bar{x}_0 = 0$ and end at $\bar{x}_1 = 2\pi n$. These are now paths in $\mathbb{R}$ with the same endpoints. But $\mathbb{R}$ is simply connected (because it is contractible), so there exists a homotopy connecting $\bar{\alpha}$ and $\bar{\alpha}'$, i.e., a map $\bar{F} : I \times I \to \mathbb{R}$ such that

deformation param.

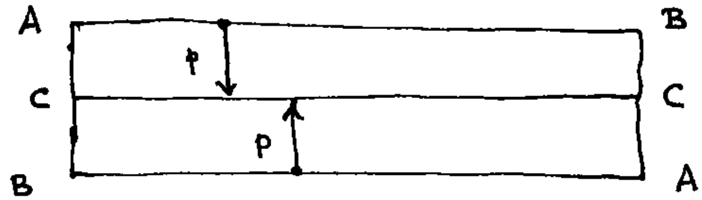
$$\bar{F}(s,0) = \bar{\alpha}(s) \qquad \bar{F}(0,t) = 0$$
$$\bar{F}(s,1) = \bar{\alpha}'(s) \qquad \bar{F}(1,t) = 2\pi n$$

Then we define $F : I \times I \to S^1$ by $F = p \circ \bar{F}$, and F is a homotopy ~~maps~~ deforming α into α'.

Finally, it is easy to show that for every n, there exists a loop α with winding number n ~~[illegible]~~. ($\alpha(s) = e^{2\pi i n s}$.) Therefore,

$$\boxed{\pi_1(S^1) = \mathbb{Z}.}$$

Notice that we used the (known) fact that $\mathbb{R}$ is simply connected. $\mathbb{R}$ is the <u>universal</u> <u>covering</u> <u>space</u> of S^1. When we unroll S^1, it is not necessary to go all the way to $\mathbb{R}$. We can stop with a new circle that is two, three, ... times as big as S^1. For example, the (single) edge ABA of the Möbius strip is the double cover of the central circle CC:



p^{-1} is double valued.

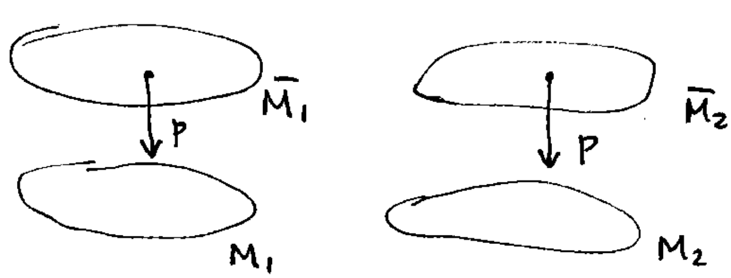
Both $ABA = S^1$ and $CC = S^1$ topologically speaking, but ABA is the <u>double cover</u> of CC. The projection p is indicated by arrows.

$$p: ABA \to CC \quad \text{or} \quad p: S^1 \to S^1.$$

Similarly, the Klein bottle possesses a double cover in the 2-torus T^2, $p: T^2 \to \text{Klein}$, p^{-1} double valued.

(official) <u>Def.</u> Let $M, \bar{M}$ be connected topological spaces with a map $p: \bar{M} \to M$ such that (1) p is surjective; and (2) for each $x \in M$ $\exists$ a connected open neighborhood $U \in M$ containing x such that $p^{-1}(U)$ is a disjoint union of open sets $\{U_\alpha\}$ in $\bar{M}$, each mapped homeomorphically onto U by p ($p(U_\alpha) = U$, $\forall \alpha$). Then $\bar{M}$ is the <u>cover</u> of M. If $\bar{M}$ is simply connected, then it is the <u>universal cover</u> of M.

<u>Remarks</u>. We require M to be connected, because otherwise we might as well talk about the cover of each component (piece) of M separately:



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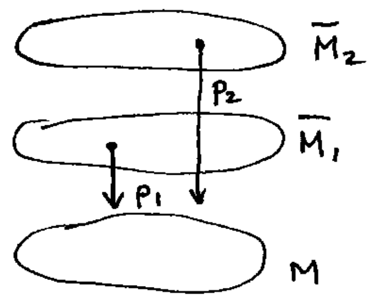
And we require $\bar{M}$ to be connected, because otherwise M can be thought of as having more than one cover, each of which can be treated separately:



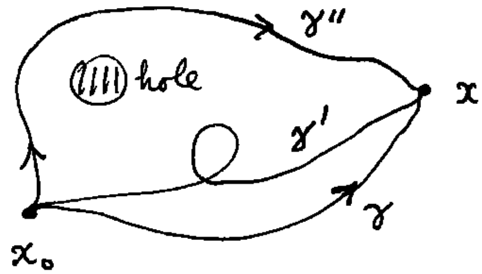
Thm: Given a contin path $\alpha : I \to M$ with $\alpha(0) = x_0 \in M$, and a choice of a point $\bar{x}_0$ in the preimage $p^{-1}(x_0)$, then exists a unique continuous path $\bar{\alpha} : I \to \bar{M}$ such that $\bar{\alpha}(0) = \bar{x}_0$ and $\alpha = p \circ \bar{\alpha}$. $\bar{\alpha}$ is the lift of α.

It's obvious that a circle can be unrolled into a line, but in fact any connected (but not simply connected) space can be "unrolled" into a connected and simply connected space (the universal cover). Given a connected M, here is how we construct the U.C.S.:

Choose $x_0 \in M$. Let (x, γ) be a (~~pair~~ point, path) pair, where $\gamma : [0,1] \to M$, $\gamma(0) = x_0$ and $\gamma(1) = x$. (γ is continuous.) It is redundant to write (x, γ), since $x = \gamma(1)$, but it is notationally convenient. Then a point of $\bar{M}$ (the U.C.S.) is an equivalence class $[(x, \gamma)]$, where

$(x, \gamma) \sim (x', \gamma')$ if $x = x'$ and γ homotopic to γ.

(M)



$[(x,\gamma)] = [(x,\gamma')]$ = one pt. of $\bar{M}$

$[(x,\gamma'')]$ = a different pt. of $\bar{M}$

So, $[(x,\gamma)]$ is a family (an equivalence class) of homotopic curves connecting x_0 and x. Since there is only a discrete set of ~~homotopy~~ classes of such homotopic curves for given x, a point $\bar{x} = [(x,\gamma)] \in \bar{M}$ is specified by x plus a discrete label for the class of curves. We define

$$p : \bar{M} \to M : [(x,\gamma)] \mapsto x,$$

so p just throws away the class information about the curve γ. (We will see in a moment how the class of equivalent curves can be specified.) Notice that if M is simply connected, then there is only one class of curves γ for given endpoints, and $\bar{M} = M$.

Consider special case that $x = x_0$. Then curve γ is a loop based at x_0, and the equivalence classes of curves γ are the same as elements of the homotopy group $\pi_1(M,x_0) \equiv G$. Write g for elements of g, which are otherwise classes $[\gamma]$ of loops γ based at x_0. Thus when $x = x_0$, the "branches" of p^{-1} are labelled by $g \in G$.

Now consider a simply connected region U containing x_0, and let $x \in U$. Then we can label the branches of p^{-1} (again) by elements of $G = \pi_1(M,x_0)$, in a continuous manner over U. First, for each point $x \in U$, we draw a conventional ~~curve~~ path τ_x from x_0 to x in U. Since U is simply connected, any other τ_x' (also in U) is homotopic

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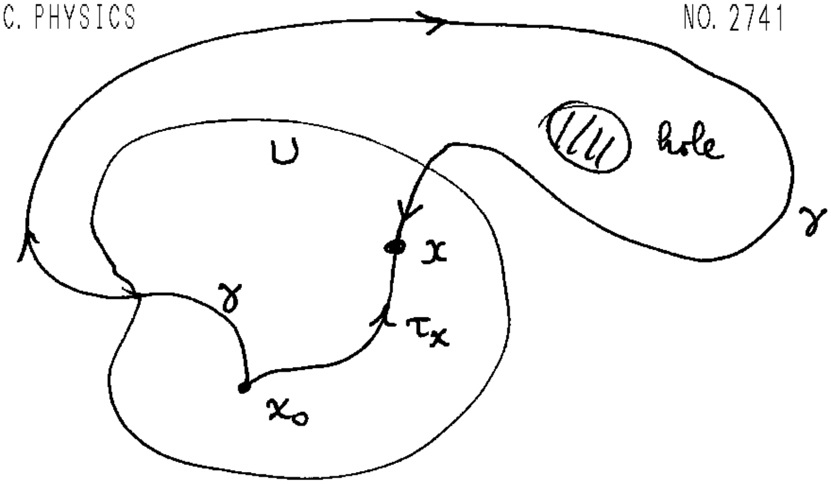
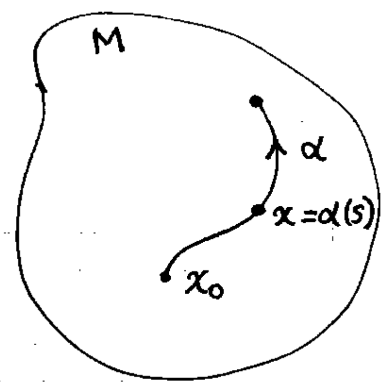
to τ_x.



Now we label the classes of curves γ that go from x_0 to x by the homotopy class $[\gamma * \tau_x^{-1}]$, that is, an element $g \in G$ ($\gamma * \tau_x^{-1}$ is a loop based at x_0). Note that γ need not be confined to U. So all points $\bar{x} = [(x, \gamma)]$ over U (that is, with $x = p(\bar{x}) \in U$) can be labelled by (x, g) where $g \in G$, and we see that locally (that is, $p^{-1}(U)$) $\bar{M}$ looks like $U \times G$. $\bar{M}$ can be thought of as a fiber <u>bundle</u> over M with a <u>discrete</u> <u>fiber</u> G.

~~Similarly, we can label points along a curve~~

Similarly, let $\alpha : I \to M$, $\alpha(0) = x_0$ be a path on M starting at x_0, and suppose it does not intersect itself.



We wish to find a convenient labelling for the discrete choices for $p^{-1}(x)$ where $x = \alpha(s)$ is a point on the path. At $x = x_0$ $(s=0)$ we just use $g \in G = \pi_1(M, x_0)$ as before. For other points we proceed as above

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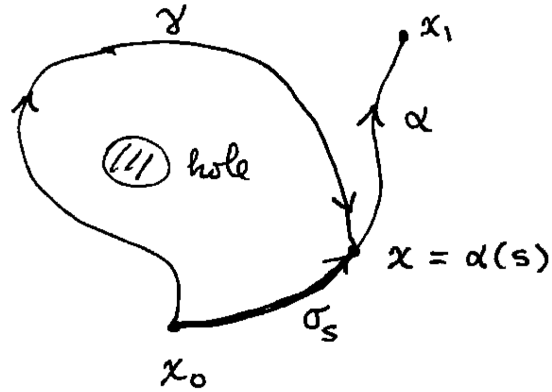
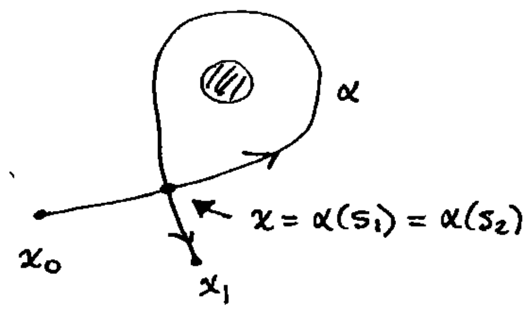
and write

$$g = [\gamma * \sigma_s^{-1}],$$

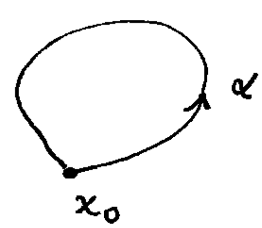
where $\sigma_s : I \to M$ is the path that covers the part of α starting at x_0 and going to $\alpha(s)$, i.e., $\sigma_s(t) = \alpha(st)$.



This labels the branches of p^{-1} along α in a continuous manner using elements $g \in G$. We do not let α cross itself because if we do, there will be two labellings of branches of p^{-1} at a single point that need not agree.



Let's examine this disagreement in the special case that α is a loop based at x_0. $(x_1 = x_0)$. Let us choose a branch of p^{-1} at x_0 labelled by $g_0 \in G$, and follow this branch continuously as we move along α from x_0 back to x_0 again.



The resulting path $\bar{\alpha}(s) = [(\alpha(s), \gamma_s)]$ in $\bar{M}$ is the <u>lift</u> of of $\alpha(s)$, where $\gamma_s(0) = x_0$, $\gamma_s(1) = \alpha(s)$, and $\gamma_s * \sigma_s^{-1}$

belongs to the equivalence class $g_0 \in \pi_1(M, x_0)$, that is,

$$\gamma_s * \sigma_s^{-1} \sim \gamma_0$$

Now setting $s=1$, $\sigma_s = \sigma_1 = \alpha$, $\gamma_s = \gamma_1$, and we get

$$\gamma_1 * \alpha^{-1} \sim \gamma_0, \qquad \gamma_1 \sim \gamma_0 * \alpha,$$

or

$$g_1 = g_0 * [\alpha].$$

Thus the branch we return on is g_1, if we left on branch g_0. ~~Equivalently, we left on branch (x_0, g_0) and retu~~ Equivalently, the lifted path $\bar{\alpha}$ starts at point $\bar{\alpha}(0) = (x_0, g_0)$ and ends at point $\bar{\alpha}(1) = (x_0, g_1)$ (generally on a different branch). From the eqn above, in fact, we see that $\bar{\alpha}$ returns on the same branch it started out on if and only if α (the projected curve) belongs to the trivial class (α is contractible).

Thus, the lift $\bar{\alpha}$ of a loop α in M is itself a loop (in $\bar{M}$) if and only if α is contractible. But as α smoothly contracts to a point, so does its lift $\bar{\alpha}$. Thus, all loops in $\bar{M}$ are contractible, and M is <u>simply connected</u>.

One more remark, if M is a <u>group</u> then the U.C.S. $\bar{M}$ can also be given the structure of a group. It is the <u>universal</u> <u>covering</u> <u>group</u> and it is simply connected. This is the relation between $SO(3)$ and $SU(2)$, and $S^1 = U(1)$ and $\mathbb{R}$.