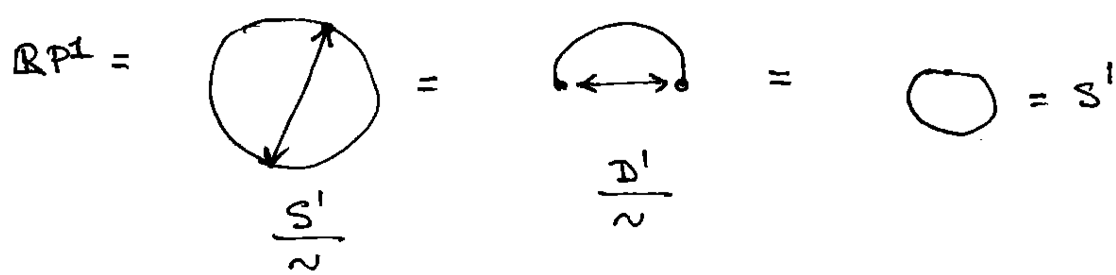
This $\mathbb{Z}_2$ is responsible for the line defect (vortex) in a nematic liquid, that can annihilate when it meets another such line defect, leaving behind no defect at all. [It's not that vortices of "opposite charge" are annihilating, rather, there is only one charge and it obeys the rule $1+1=0$.]

While on the subject of $\mathbb{RP}^n$, note special case $n=1$. Recall, $\mathbb{RP}^n$ is the sphere S^n with antipodal points identified; equivalently, it is the n-dimensional disk D^n (= the "northern hemisphere" of S^n) with antipodal points on the boundary identified. (The disk D^n is region $r \le 1$ in n-dim. space $\mathbb{R}^n$; it is the sphere S^{n-1} plus all interior points.) So, for $n=1$, we get a circle with opposite points identified, or D^1, the 1-disk, which is a line segment with ~~opposite~~ endpoints identified:



You might say the final circle is ½ as big as the first one.

Now is a good time to comment on the relationship between classical rotations and spin rotations in QM. A classical rotation is an element of $SO(3)$, a linear map of $\mathbb{R}^3$ onto itself that preserves lengths, angles, and cross products. It takes 3 parameters to specify some $R \in SO(3)$ (i.e., $SO(3)$ is a 3-dimensional manifold). These parameters can be specified in various ways. One is the Euler angles (often an ugly choice). Another is the <u>axis-angle</u> parameterization:

(continued on p. 11).

2/12/02

<u>Motivation</u> for studying relationship between SO(3) and SU(2). Consider evolution of spin 1/2 particle in magnetic field (neutron maybe) $\vec{B}=\vec{B}(t)$ which we allow to be time-dep. Define

$$\vec{\omega}(t) = g\,\frac{e}{2mc}\,\vec{B}(t)$$

a vector with dimensions of frequency (g = g-factor of particle).
Let $\chi = \begin{pmatrix}\chi_\uparrow \\ \chi_\downarrow\end{pmatrix} \in \mathbb{C}^2$ be the usual spinor. The Schrödinger eqn is

$$i\hbar\frac{\partial\chi}{\partial t} = \vec{\omega}(t)\cdot\left(\frac{\hbar}{2}\vec{\sigma}\right)\chi \qquad \text{(Qu)}$$

where $\frac{\hbar}{2}\vec{\sigma}$ is the spin operator. Let $\vec{S}(t)$ be the expectation value of the spin operator,

$$\vec{S}(t) = \langle\chi(t)|\,\frac{\hbar}{2}\vec{\sigma}\,|\chi(t)\rangle$$

so that $\vec{S}$ is a c-number vector (not a vector of operators, $\vec{S}\in\mathbb{R}^3$).
Then

$$\frac{d\vec{S}}{dt} = \vec{\omega}(t)\times\vec{S} \qquad \text{(Cl)}.$$

(Qu) is the "quantum eqn" and (Cl) is the "classical" eqn. (classical in the sense that eqns just like this occur in classical mechanics, they are the Euler equations). The solutions of (Qu) and (Cl) are

$$\chi(t) = U(t)\,\chi_0, \qquad U(t)\in SU(2)$$

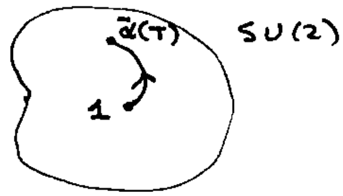
$$\vec{S}(t) = R(t)\,\vec{S}_0, \qquad R(t)\in SO(3).$$

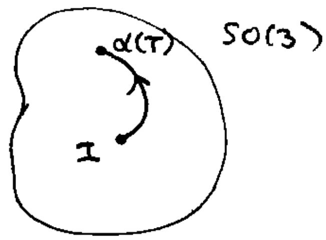
where $U(0)=1$ (the 2×2 identity) and $R(t)=I$ (the 3×3 identity).
The functions $U(t)$ and $R(t)$ are actually <u>paths</u> on the group manifolds SU(2) and SO(3). Let

(T = final time)

$$\alpha : [0,T] \to SO(3)$$
$$\bar{\alpha} : [0,T] \to SU(2)$$

be two paths in SO(3) and SU(2), where $\alpha(t)$ means $R(t)$ and $\tilde{\alpha}(t)$ means $U(t)$, satisfying $\tilde{\alpha}(0) = 1$, $\alpha(0) = I$. Picture on the group manifolds,





Consider the stmt: "If you rotate a neutron by 360°, it doesn't return to its original self but rather undergoes a phase change of −1. You have to rotate it by 720° to make it return to itself." Actually it is not the <u>final</u> <u>value</u> of the classical rotation $R(t)$ (or $\alpha(t)$) that determines the outcome, but rather the <u>history</u>. Here is a correct stmt:

Let $R(T) = \alpha(T) = I$ (at $t=T$). Then $\alpha : [0,T] \to SO(3)$ is a loop based at I. But $SO(3) = \mathbb{RP}^3$ (topologically speaking), so there are two homotopy classes the loop α can be in, the trivial class or the nontrivial class, since $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$. Then

$$U(T) = \tilde{\alpha}(T) = \begin{cases} +1 & \text{if } \alpha \in \text{trivial (contractible) class} \\ -1 & \text{if } \alpha \in \text{other class.} \end{cases}$$

The final state of the neutron depends on the homotopy class of the loop α in SO(3). In fact one may say that the existence of spin is related to this nontrivial homotopy group $\pi_1(SO(3)) = \mathbb{Z}_2$.

2/12/02

There is an important map $p: SU(2) \to SO(3)$ that occurs in this theory. (p stands for "projection.") It is defined by ...

$$R_{ij} = \frac{1}{2} \mathrm{tr}\,(U^{+} \sigma_i U \sigma_j) \qquad \text{where } U \in SU(2),$$

i.e., it defines a function $R(U)$ or $R = p(U)$. One can show that

$$R(t) = p(U(t))$$

in the spin problem, i.e., p maps the path $\bar{\alpha}(t)$ in $SU(2)$ into $\alpha(t)$ in $SO(3)$. Note that $p(U) = p(-U)$, so the inverse $p^{-1}(R)$ of $R \in SO(3)$ consists of 2 points U and $-U$ (it turns out there are only these two). p is a two-to-one projection.

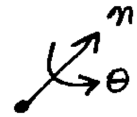
$SU(2)$ is ~~an example of~~ said to be a double cover of $SO(3)$. This is an example of a space M (here $SO(3)$) and its covering space $\bar{M}$ (here $SU(2)$). The projection p in the general case ~~has~~ is a map $p: \bar{M} \to M$ from the covering to the covered spaces. The path $\bar{\alpha}(t)$ defined above in $\bar{M} = SU(2)$ is called the lift of the path $\alpha(t) = R(t)$ in $M = SO(3)$. We mention all this (as yet) undefined terminology to give an example a preview of what will come.

Covering spaces don't have to be groups, but in this example they are, and there is extra structure because of that. For example, $p: SU(2) \to SO(3)$ is a group homomorphism, with kernel $\{1, -1\}$ (the image is all of $SO(3)$).

2/12/04

Take it as geometrically obvious that an arbitrary (proper) rotation can be written in axis-angle form:

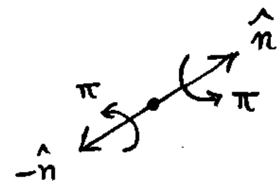
$\hat{n} \in S^2$, $0 \le \theta \le \pi$

$R(\hat{n},\theta) =$ 

using right hand rule.

The parameterization is unique except when $\theta = 0$, where $R(\hat{n},0) = I$ for any $\hat{n}$, and at $\theta = \pi$, where

$$R(\hat{n},\pi) = R(-\hat{n},\pi)$$



(same rotn)

So if we write $\vec{\theta} = \hat{n}\theta$, so that $\vec{\theta} \in \mathbb{R}^3$, then $SO(3)$ is identified with a sphere (the 3D, solid interior of a sphere in $\mathbb{R}^3$) out to a radius of π, including the surface (S^2) at $\theta = \pi$, but with antipodal points $(\hat{n}, -\hat{n})$ on the surface identified. In other words,

$$SO(3) = \mathbb{R}P^3.$$

This is an example of a group manifold.

As for $SU(2)$, $\rightarrow$ physically, spin rotations. it is the set of 2×2, complex, unitary matrices with $\det = +1$:

$$U \in SU(2) \quad \Rightarrow \quad UU^+ = U^+U = I \text{ and } \det U = +1.$$

The condition $UU^+ = U^+U = I$ means that the rows and columns form pairs of orthonormal, complex, unit vectors (in $\mathbb{C}^2$). Write

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a,b,c,d \in \mathbb{C},$$

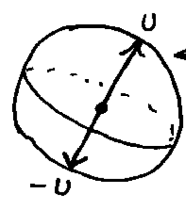
Because of the conditions $UU^{+} = I$, $\det U = +1$, the 4 complex components of $U \in SU(2)$ satisfy certain constraints, and U can be written in terms of 4 real parameters $(x_0, x_1, x_2, x_3) = (x_0, \vec{x})$,

$$U = x_0 I - i\,\vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix}$$

where $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. The (x_0, x_1, x_2, x_3) are the <u>Cayley-Klein parameters</u>, and they show that topologically

$$SU(2) = S^3.$$

The relation between $SU(2) = S^3$ and $SO(3) = \mathbb{RP}^3$ is just the identification of antipodal points U and $-U$ in S^3 with a single element of $SO(3) = \mathbb{RP}^3$:



surface supposed to represent S^3 in $\mathbb{R}^4$

So $SO(3)$ can be thought of as the "northern hemisphere" of S^3 with antipodal points on the "equator" (S^2) identified. This is the solid ball picture of $SO(3)$ (in $\vec{\theta}$ coordinates).

~~$SU(2)$ is said to be a "double cover" of $SO(3)$. This is an example of a covering space. Here is another, simpler, example.~~

~~Let $p: \mathbb{R} \to S^1$ (p = "projection") be the map defined by $p(x) = e^{ix}$, where S^1 is identified with the unit circle in the complex plane. $\mathbb{R}$ can be seen as "wrapping around" S^1.~~