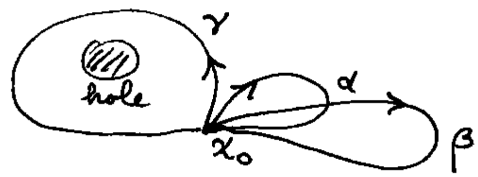
2/12/04

Review: We are thinking of maps $f: S^1 \to M$ of the circle onto a given space M, in order to understand the topology of M and for other reasons. We usually represent these maps in a different form, $f: I \to M$ where $I = [0,1]$ is the unit interval and where we require $f(0) = f(1) \equiv x_0 \in M$. We call such a map a loop based at x_0.

Loops are considered equivalent (homotopic) if they can be continuously deformed into one another. The map that does the deformation is called the homotopy.



In the picture, $\alpha \sim \beta$ but α not $\sim \gamma$ (all loops based at x_0).
The set of equivalence classes of loops based at a point x_0 can be given a group structure, in which the multiplication is just catenation. The group is denoted $\pi_1(M, x_0)$, the first homotopy group based at x_0. The group $\pi_1(M, x_1)$ based at a different point is isomorphic to $\pi_1(M, x_0)$, if M is connected. Thus, as abstract groups, they are the same. This group is denoted $\pi_1(M)$, the fundamental group.

2/12/04

Now we explore some of the properties of the fundamental group $\pi_1(M)$ of a manifold M, and work on ways of computing it.

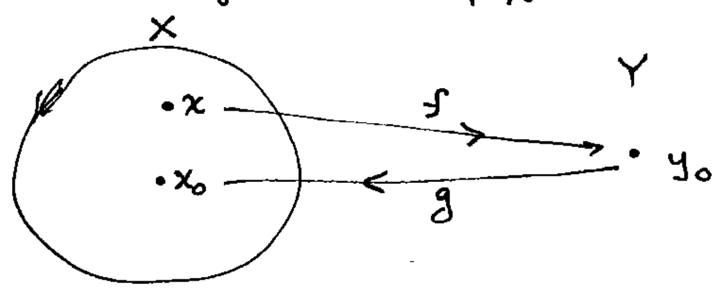
Recall defn of homeomorphism: Let X, Y be topological spaces. Then $f: X \to Y$ is a homeomorphism if $\exists$ contin maps $f: X \to Y$, $g: Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

Now we introduce a weaker concept. Let X, Y be topological spaces.

Def. X and Y are of the same homotopy type if $\exists$ contin. maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$. ($\sim$ means, "is homotopic to"). Point is that = for a homeomorphism is replaced by $\sim$ for same homotopy type.

An example of spaces that are of same homotopy type but not homeomorphic. Let $X =$ 2-disk D^2, $Y =$ single point y_0. Let $f: X \to Y$ be the const. map $f(x) = y_0$, $\forall x \in X$. Let $g: Y \to X$ map y_0 onto a certain point $x_0 \in X$.

center of disk.



Not homeomorphic, because f not invertible. ~~Also~~ Are X, Y of same homotopy type? Well, $f \circ g: Y \to Y: y_0 \mapsto y_0$ is just the identity map id_Y (has to be, since Y only has one point.). Next $g \circ f: X \to X$: x (any x) $\mapsto x_0$, it is the constant map. So is $g \circ f \sim id_X$? Yes, just shrink disk by radial factor, to make disk $\to$ central point.

Notice that with "same homotopy type" (as with "homeomorphic") we don't talk about deforming <u>spaces</u> (which would require an imbedding space), we talk about deforming <u>maps</u>.

Other examples of same homotopy type spaces:

X	Y
$\mathbb{R}^n$	$\{\text{one point}\}$

annulus — circle S^1

← These examples show that being of "same homotopy type" allows dimensions of spaces X, Y to be different, no invertible map

If sets are homeomorphic, then they are of same homotopy type (but not the converse).

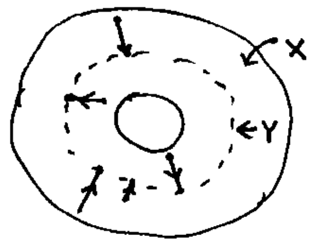
<u>Fact</u>: "Same homotopy type" is an equivalence relation.

<u>Thm</u>: If X and Y are of the same homotopy type, then $\pi_1(X) = \pi_1(Y)$.

We won't prove this thm, but the proof is not very hard, and if you try it you will see the motivation for the definition of "same homotopy type". In any case, this theorem implies that homotopy groups are <u>topological invariants</u>.

So how to tell if spaces are of same homotopy type? May not be easy, but one case arises a lot in practice, occurs when Y is a subset of X. Suppose we have a family of continuous maps parameterized by a deformation parameter $t \in [0,1]$ which is the identity at $t=0$, which maps X onto Y when $t=1$, and which leaves points of Y alone for all t.

Picture, example: X = annulus
Y = circle (dotted)



shown are paths of a point $x \in X$ under the map, as deformation parameter goes from 0 to 1.

Official definition: Let X, Y be topological spaces, $Y \subset X$.
A deformation retract is a map $F : X \times [0,1] \to X$ such that

$F(x,0) = x$ (identity @ $t=0$)
$F(x,1) \in Y$ (into Y at $t=1$)

$F(y,t) = y, \quad \forall t \in [0,1]$ (Y invariant, all t).

Fact: If ~~X~~ Y is a deformation retract of X, then X, Y are of same homotopy type. [$f : X \to Y$ is the retraction at $t=1$, $g : Y \to X$ is the inclusion.]

Another Def. If $x_0 \in X$ is the deformation retract of X (special case $Y = \{x_0\}$ = one point), then X is contractible.

Follows immediately,

Corollary: If X is contractible, then $\pi_1(X) = \{e\}$ (the trivial group).

Def. If $\pi_1(X) = \{e\}$, then X is simply connected.

Before going on, let's get some <u>examples</u> of fundamental groups, obtained by intuition if nothing else.

(1) First, $\pi_1(\mathbb{R}^n) = \{e\}$ (the trivial group), because all loops are contractible (the space is simply connected). This is obvious by drawing pictures,

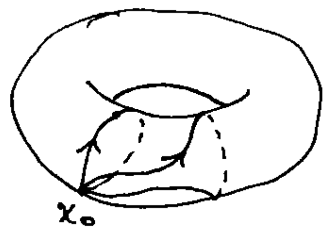
(1a) Note the special case $n=0$, $\mathbb{R}^0$ = one point $= \{0\}$, $\pi_1(\underset{\text{space}}{\text{one point}}) = \{e\}$.

(2) Next, $\pi_1(S^1) = \mathbb{Z}$, where the integer $n \in \mathbb{Z}$ is the "winding number" of the map. This is intuitively clear if you wrap a rubber band around a cylinder. We consider a more formal argument below.

(3) Next, $\pi_1(S^n) = \{e\}$ for $n > 1$. This is intuitively obvious for S^2 (any closed loop on S^2 can be contracted to a point):

and a similar logic works for S^n for higher n.

(4) Next, $\pi_1(T^2) = \mathbb{Z}^2$ (the 2-torus), as is intuitively clear by drawing pictures,



The homotopy class is determined by the two "winding numbers".

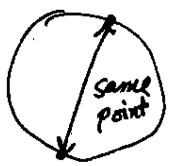
2/12/02

This can be proved from the case $\pi_1(S^1) = \mathbb{Z}$ by using the theorem that the fundamental group of the Cartesian product of two arcwise connected spaces is the Cartesian products of the groups,

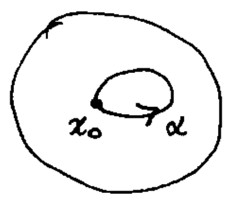
$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

The same thing shows that $\pi_1(T^n) = \mathbb{Z}^n$ (n winding numbers on an n-torus), since $T^n = S^1 \times \ldots \times S^1$ (n times).
(also mention cylinder $= \mathbb{R} \times S^1$).

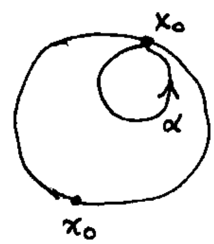
(5) Take the case of $\mathbb{R}P^2$, ~~circle~~ disk with opposite bdry points identified.



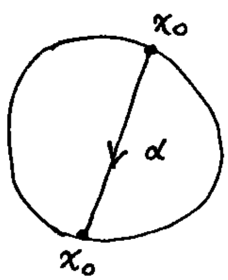
Easy to find contractible ~~curves~~ loops:



or

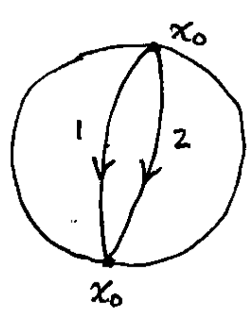


Here is a ~~curve~~ loop that is not contractible:

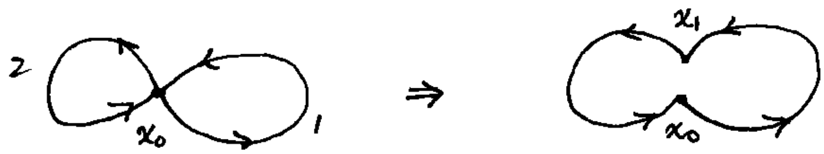


Because you can't bring the two attachment points together (they stay on opposite sides of the bdry under any contin. deformation).

So $[\alpha]$ is a nontrivial element of $\pi_1(\mathbb{R}P^2, x_0)$. Now look at $[\alpha] * [\alpha]$, equiv. class of loops that ~~get~~ are homotopic to traversing α twice. Label the 2 traversals 1, 2 to indicate order, and bow them out slightly to separate them:

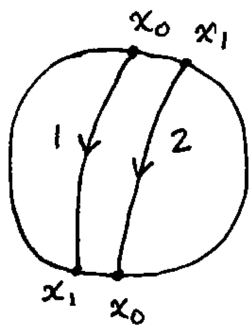


This path starts at x_0, passes through x_0 a 2nd time, then comes back to x_0 a 3rd time. Pull the path away from x_0 on the second encounter, like this:

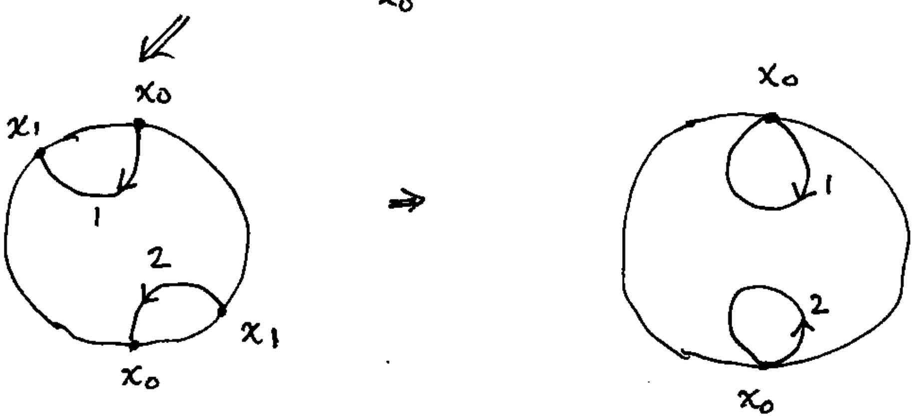


still (leaves a loop based at x_0)

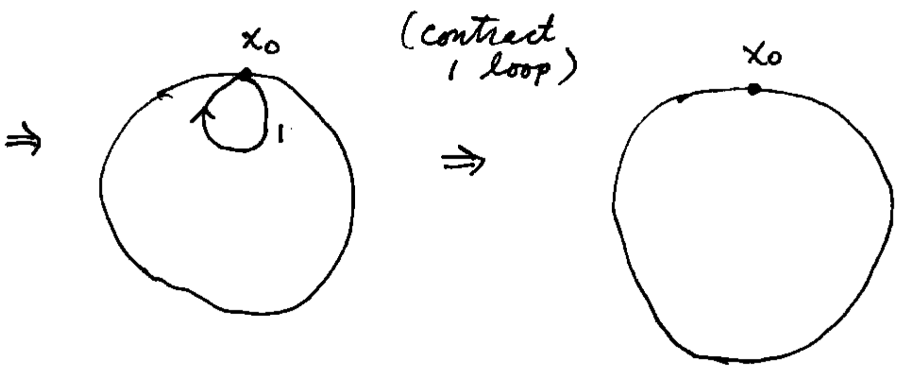
which on $\mathbb{R}P^2$ looks like this:



Then deform further by moving x_1 around to meet x_0:



(contract 2 loop)



(trivial loop at x_0).

Thus $[\alpha] * [\alpha] = [c]$ = trivial class, contractible.

Hence $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$. A similar argument shows that

$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$, $n \geq 2$.