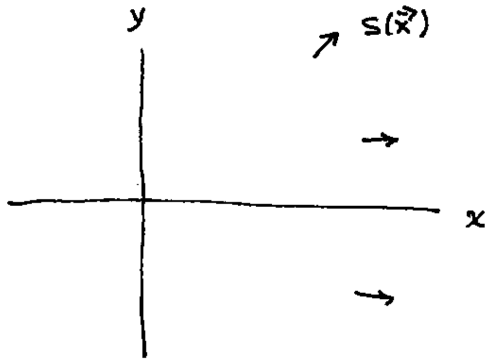
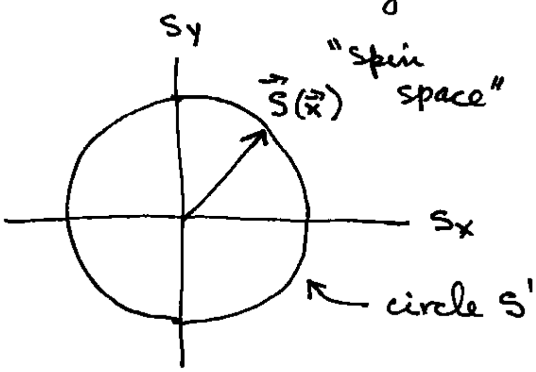
2/10/04

Begin with motivation for homotopy theory from CM physics. (But $\exists$ many other applications.) Consider a 2D model of a Heisenberg ferromagnet, treated as a continuum, so we have a spin vector $\vec{S}(\vec{x})$ whose magnitude is fixed, $|\vec{S}| = S = \text{const}$. Assume $\vec{S}$ lies in the plane.

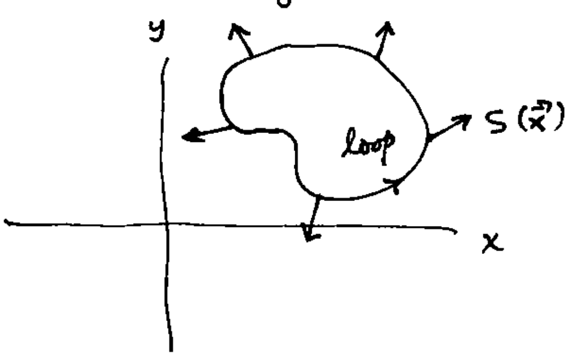


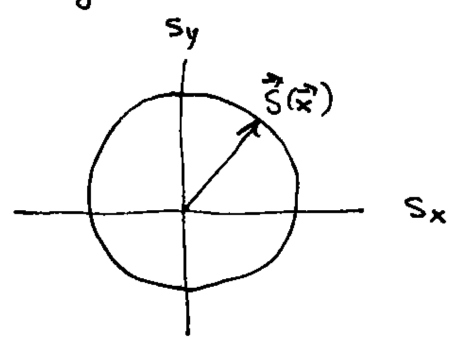
Can regard the spin field as a map $f: X \to S^1$, where X is the region of the plane occupied by the ferromagnet (maybe X = all of $\mathbb{R}^2$, depends on the model). The ~~domain~~ range is S^1 (the circle) because only the direction of $\vec{S}$ can vary, not the magnitude.



The same picture would apply to any 2D vector field in the plane (or some region of it), as long as the vector field did not vanish, (that is, you could look at the direction of the vector field).

Now consider a closed loop in the plane. As we go around the loop, the vector $\vec{S}(\vec{x})$ goes around its circle, returning to where it started.





The number of times $\vec{S}(\vec{x})$ goes around its circle as $\vec{x}$ goes around its loop is the <u>winding number</u> of the field around the loop.

Note that if $\vec{S}(\vec{x}) = \text{const}$ (indep. of $\vec{x}$) then the winding # is 0.

Now we can prove that if the winding # is not 0, then there must be a singularity inside the loop. Suppose not. Then we can continuously contract the loop until it is very small. Since we are assuming $\vec{S}$ is continuous (this is what we mean by singularity-free) then over a small loop $\vec{S}$ is nearly constant and the winding # is zero. But the winding number is an integer and cannot change discontinuously. ● So there must be a discontinuity inside the loop.

This would be called a point defect in the plane in CM terminology. The loop provides a map $f: S^1 \to S^1$, where the first S^1 is the loop in the plane and the second is in spin space. The winding number is a characteristic of the map of the circle to itself, also called the Brouwer degree of the map. This number, being an integer, cannot change under continuous deformations of either the loop or the field $\vec{S}(\vec{x})$ itself. Such changes can be thought of as inducing continuous changes of the map $f: S^1 \to S^1$. Homotopy theory studies invariants of maps such as this one under continuous changes.

The range of f (S^1 in this case) is called "order parameter space" in CM applications. Another problem with the same order parameter space is superfluid ^4He, where the superfluid is described by a field,

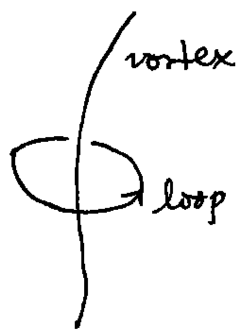
$$\psi(\vec{x}) = A e^{i\varphi}$$

(ψ is an expectation value of a quantum field, thus it is a classical, c-number complex field on $\mathbb{R}^3$ or some domain $X \subset \mathbb{R}^3$).

The amplitude A is related to the superfluid density which normally is nonzero throughout the volume. The phase φ is related to the fluid velocity via

$$\vec{v} = \frac{\hbar}{m} \nabla \varphi .$$

The phase specifies a point on a circle $e^{i\varphi}$ in the complex plane, so again we may regard order parameter space as a circle and think of a map $f: X \to S^1$ where now $X \subset \mathbb{R}^3$. Or if we restrict f to a loop in X we get a map $f: S^1 \to S^1$ as before, with a winding number. Again, the logic shows that if the winding number is nonzero, then the ~~circle~~ loop must enclose a singularity. This time, however, the domain is in $\mathbb{R}^3$ so the singularity is a line, not a point. It is a line defect, in this case, called a vortex.



$f: \text{loop} \to S^1$ has winding number

A similar situation occurs in superconductors, when the Cooper-paired spins are in the singlet state (hence Ψ describes a spin 0 boson, like ^{4}He).

A different type of order parameter space occurs with nematic liquids. These are liquids with long molecules that behave like rigid rods, which tend to align with their neighbors. The aligment is not described by a vector field, however, because the properties of the nematic are the same in both directions, like a double-headed object $\longleftrightarrow$.

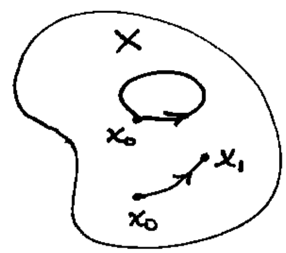
Thus the order parameter space is $\mathbb{RP}^2$. Nematics support line defects that are studied by examining continuous deformations of maps $f: S^1 \to \mathbb{RP}^2$ (S^1 because a circle goes around a line defect). It turns out that nematics also support point defects that are studied by surrounding the point with a closed surface, i.e., one homeomorphic to S^2, and studying invariants of maps $f: S^2 \to \mathbb{RP}^2$.

There are other kinds of order parameter spaces. For spins in 3D, the spin vector is identified with a point on S^2. There are no line defects, but point defects are possible and lead us to study maps: $S^2 \to S^2$. Another example is superfluid ^{3}He in the dipole locked phase, where the order parameter can be thought of as two vectors (nonzero), locked to be orthogonal ↳ to one another. The orientation of such an object is specified relative to a standard orientation, by means of a rotation $\in SO(3)$. Hence order parameter space is $SO(3)$, which is homeo. to $\mathbb{RP}^3$ (as it turns out).

To study defects we have looked at maps from $\underset{\text{line}}{S^1}$ or $\underset{\text{point (in } \mathbb{R}^3)}{S^2}$ to order parameter space ($S^1, \mathbb{RP}^2, S^2, SO(3), \ldots$). A map from S^3 to order parameter space occurs when we want to study <u>singularity-free</u> field configurations ~~that~~ in $\mathbb{R}^3$ that satisfy the constraint of having a fixed asymptotic value as $r \to \infty$. We just "compactify" $\mathbb{R}^3$ into S^3 by introducing the "point at ∞" (where the field has a fixed value), and ~~so~~ study maps $f: S^3 \to$ O.P.S. such that f(pt. at ∞) = given value. For example, the constant map is one such field configuration, but (depending on the O.P.S.) there may be others which cannot be reached by continuous deformation from the constant case. Again we are interested in properties of maps $f: S^n \to M$ (some manifold = O.P.S.) that are invariant under contin. deformations.

We take the preceding as motivation for considering continuous deformations of maps: $S^n \to M$ = some manifold. First take case S^1.

Let $I = [0,1]$.

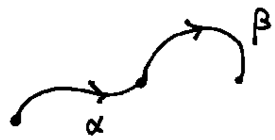


A <u>path</u> is a map $f: I \to X$ = some topological space

A <u>loop</u> is a path such that $f(0) = f(1) = x_0$ = "<u>base point</u>"

Note, path can self-intersect, or even be just a point (the constant path).

Now, basic properties of paths and loops. First, we can <u>multiply</u> paths if the endpoints match. This is just <u>catenation</u>.



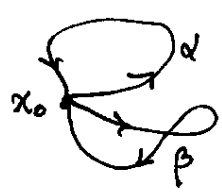
$= \alpha * \beta$.

Let $\alpha, \beta : [0,1] \to X$, let $\alpha(1) = \beta(0)$. Define:

$$(\alpha * \beta)(s) = \left\{ \begin{array}{ll} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{array} \right\}.$$

Notice order: traverse α first, β second.

Note that loops based at a common point can always be multiplied.



$\alpha * \beta$ meaningful.

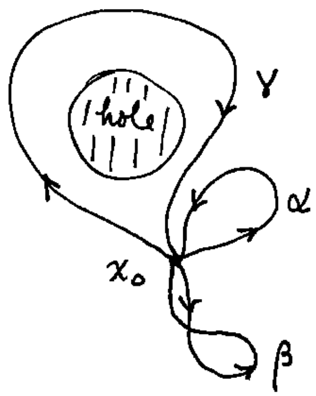
2/10/04

Def: Let $\alpha : [0,1] \to X$ be a path. Define the inverse path by

$$\alpha^{-1} : [0,1] \to X, \qquad \alpha^{-1}(s) = \alpha(1-s).$$

The inverse path just traverses the original path in the reverse order.

Now we define homotopic equivalence, specializing to the case of ~~and~~ loops based at a point. Idea:



α is homotopically equivalent to β, but not to γ (they cannot be continuously deformed into one another).

Let $\alpha, \beta : \underbrace{[0,1]}_{=I} \to X$ be loops based at $x_0 = \alpha(0) = \alpha(1) = \beta(0) = \beta(1)$.
Then α is said to be homotopic to β if $\exists$ a continuous function $F : I \times I \to X$ (think, $F(s,t)$, s = parameter of path, t = deformation param.) such that

$$F(s,0) = \alpha(s) \qquad \text{and} \qquad F(0,t) = F(1,t) = x_0.$$
$$F(s,1) = \beta(s)$$

Then F (the deformation map) is said to be the homotopy.

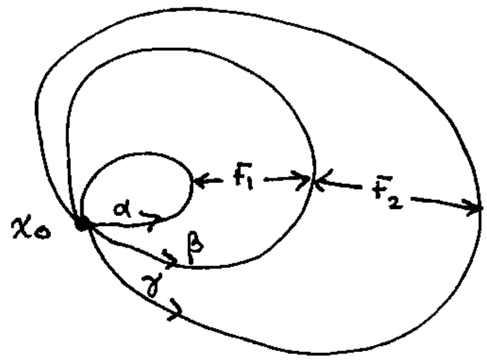
Basic fact is, "homotopic to" is an equivalence relation. To prove this, have to show 3 things:

(a) $\alpha \sim \alpha$

(b) $\alpha \sim \beta \Rightarrow \beta \sim \alpha$

(c) $\alpha \sim \beta$ and $\beta \sim \gamma \Rightarrow \alpha \sim \gamma$.

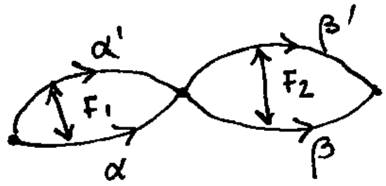
I will just prove (c), which is the hardest. A picture gives the idea:



Let $$F(s,t) = \begin{cases} F_1(s, 2t), & 0 \le t \le 1/2 \\ F_2(s, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Then F deforms α into γ.

Thus we have equivalence classes of loops based at a point x_0, $[\alpha]$, $[\beta]$, etc.
We want to define $$[\alpha] * [\beta] = [\alpha * \beta] \quad \text{(multiplication of classes).}$$
But we have to show that this is consistent (independent of choice made for representative element). That is, let $\alpha' \sim \alpha$ and $\beta' \sim \beta$. Want to show that $\alpha' * \beta' \sim \alpha * \beta$. Draw a picture, makes it obvious:



[Here picture drawn for arbitrary paths, but you get loops if endpoints are same point.]. Picture shows what to do:

Let $$F(s,t) = \left\{ \begin{array}{ll} F_1(2s,t), & 0 \le s \le 1/2 \\ F_2(2s-1,t), & \frac{1}{2} \le s \le 1 \end{array} \right\}.$$

F is the homotopy that deforms $\alpha' * \beta'$ into $\alpha * \beta$.

Turns out, this $*$ law on equivalence classes of loops based at a point x_0 defines a group. Need to show:

$$([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma]) \quad \text{associative law} \quad (\text{drop } * \text{ now})$$

$$[\alpha][c] = [c][\alpha] = [\alpha], \quad [c] = \text{equiv. class of constant } \cancel{\text{map}}\ \text{loop},$$

$$[\alpha^{-1}][\alpha] = [\alpha][\alpha^{-1}] = [c]. \qquad c: [0,1] \to X: s \mapsto x_0.$$

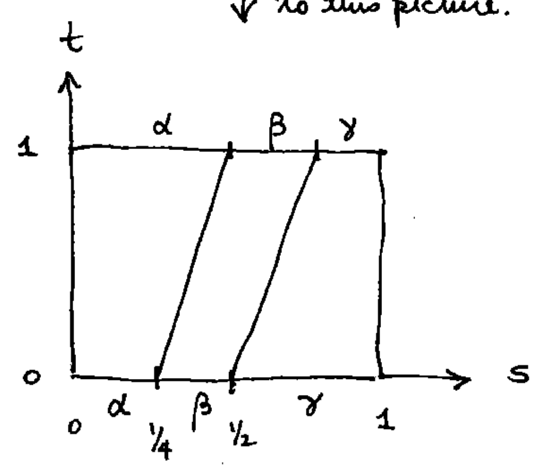
Thus $[c]$ is the identity: It is the equivalence class of loops that can be contracted to a point.

I will just prove the associative law (the others are easy). A picture makes it obvious, and also shows how to construct the formal proof:

$$(\alpha*\beta)(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1 \end{cases}$$

$$((\alpha*\beta)*\gamma)(s) = \begin{cases} \alpha(4s), & 0 \le s \le 1/4 \\ \beta(4s-1), & 1/4 \le s \le 1/2 \\ \gamma(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

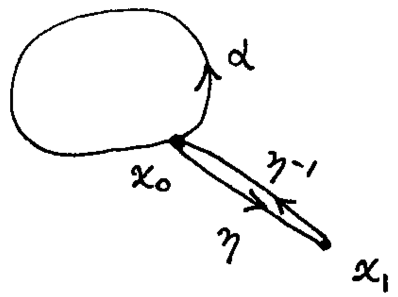
deform according to this picture.



Thus, the set of equivalence classes of loops based at a point x_0 constitutes a <u>group</u>, denoted

$$\pi_1(X, x_0) = \text{first homotopy group of } X \text{ based at } x_0.$$

This group is defined relative to a base point x_0. If we choose a different base point we get a different group, say, $\pi_1(X, x_1)$. What is the relation between these? Answer is easy in an <u>arc-wise connected</u> <u>space</u> (which for us is just a connected space), in which a path $\eta: [0,1] \to X$, $\eta(0) = x_0$, $\eta(1) = x_1$ always exists.



Given α based at x_0, can create an α' based at x_1 by writing,

$$\alpha' = \eta^{-1} * \alpha * \eta.$$

Moreover, if $\alpha \sim \beta$, then $\alpha' \sim \beta'$, so the conjugation by η preserves the equivalence class structure. (Easy to show.) So we have a mapping parameterized by η,

$$P_\eta : \pi_1(X, x_0) \to \pi_1(X, x_1) : [\alpha] \mapsto [\eta^{-1} * \alpha * \eta]$$

In fact, P_η is a group isomorphism. We prove this by showing first that P_η is a homomorphism, then that P_η^{-1} is a homomorphism.

Proof that P_η is a homomorphism is easy.

2/10/04

Need to show that $(P_\eta[\alpha])(P_\eta[\beta]) = P_\eta[\alpha\beta]$ (omit $*$)

LHS $= [\eta^{-1}\alpha\eta][\eta^{-1}\beta\eta] = [\eta^{-1}\alpha\eta\eta^{-1}\beta\eta] = [\eta^{-1}\alpha\beta\eta] =$ RHS.

Similary, consider $P_\eta^{-1} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ defined by

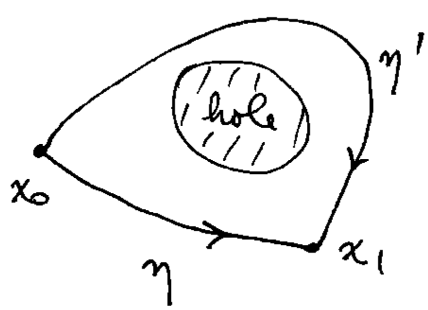
$P_\eta^{-1}[\gamma] = [\eta\gamma\eta^{-1}]$ where $\gamma \in \pi_1(X, x_1)$ and η as before.

Then easily show that P_η^{-1} actually is the inverse of P_η. Thus, P_η is a bijection, hence an isomorphism.

Thus, while $\pi_1(X, x_0)$ depends on the base point x_0, the group $\pi_1(X, x_1)$ at any other point (remember this is an arcwise-connected space) is isomorphic to it. Thus, as <u>abstract groups</u>, these groups are the <u>same</u>: it is written simply as

$\pi_1(X)$, called the <u>fundamental group of X</u>.

But notice, the isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ is not a <u>natural</u> isomorphism, because it depends on the choice of η connecting x_0 and x_1.



$P_\eta \neq P_{\eta'}$ in this picture.