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Review,

$$B_r(K) \subseteq Z_r(K) \subseteq C_r(K)$$

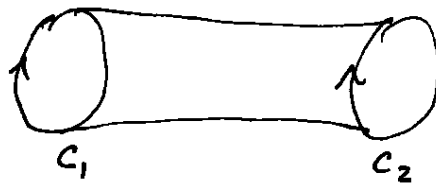
$$H_r(K) = \frac{Z_r(K)}{B_r(K)} \quad (\text{cycles modulo boundaries})$$

Special case
 $r=0$: $Z_0(K) = C_0(K)$, hence $H_0(K) = \frac{C_0(K)}{B_0(K)}$.

in fact, $H_0(K) = \mathbb{Z}^N$, $N = \#$ of connected components of K .

Special case
 $r=n = \dim K$: $B_n(K) = \{0\}$, $H_n(K) = Z_n(K)$.

A general principle (connection between homology and homotopy).
 Suppose we have two cycles that can be continuously deformed into one another ("freely homotopic"). Picture for example two 1-cycles,



As C_1 deforms into C_2 it sweeps out a 2-chain d , with $C_1 - C_2 = \partial d$. Hence C_1 and C_2 are homologous. Of course, in our approach to homology theory, the 1-cycles and the 2-chain should be made up out of the simplexes of a triangulation. But the principle is the same:

Two cycles that can be continuously deformed into one another (are freely homotopic) are homologous.

The converse however is not true, see the HW.

Now do some examples of homology groups.

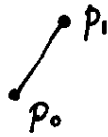
1. Single point, $\bullet p_0$, $K = \{p_0\}$.

$$H_0(K) = \mathbb{Z} \quad (\text{one connected component}).$$

$$H_r(K) = \{0\}, \quad r > 1.$$

We always understand $H_r(K) = \{0\}$ for $r > n = \dim K$.

2. A line segment.



$$K = \{(p_0 p_1), (p_0), (p_1)\}.$$

$$H_0(K) = \mathbb{Z} \quad (\text{one connected component}).$$

$$H_1(K) = \frac{Z_1(K)}{B_1(K)} = Z_1(K) \quad \text{since } B_1(K) = \{0\}.$$

So we need $Z_1(K)$. First what is $C_1(K)$? There is only one 1-simplex in K , $(p_0 p_1)$, so

$$C_1(K) = \text{gen} \{(p_0 p_1)\} = \{n(p_0 p_1) \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

All 1-chains are just integer multiples of $(p_0 p_1)$. See which ones of these are ~~nontrivial~~ cycles.

$$\partial [n(p_0 p_1)] = n \partial(p_0 p_1) = n[(p_1) - (p_0)] = 0 \quad (\text{demand}).$$

But $(p_0), (p_1)$ are independent, so this $\Rightarrow n=0$. The only 1-cycle is 0, hence $Z_1(K) = \{0\}$, and

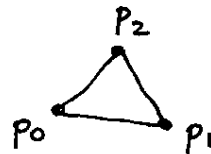
$$H_1(K) = \{0\}.$$

Beware, Nakahara writes 0 instead of $\{0\}$. Thus, for example, he would write

$$H_1(K) = \frac{\{0\}}{\{0\}} = \frac{Z_1(K)}{B_1(K)} \quad \text{in this example}$$

He would write $\frac{0}{0}$ instead of $\frac{\{0\}}{\{0\}}$. The latter (correct) expression is perfectly well defined; each group contains one element.

3. A triangle,
↑
(the boundary of)




$$n = \dim K = 1.$$

$$K = \{ (p_0 p_1), (p_1 p_2), (p_2 p_0), (p_0), (p_1), (p_2) \}.$$

Note, $|K|$ is homeomorphic to a circle S^1 .

Again, case $r=0$ is easy, $H_0(K) = \mathbb{Z}$.

For $r=1$ we have $H_1(K) = Z_1(K)$. Intuitively obvious that there is one 1-cycle, $(p_0 p_1) + (p_1 p_2) + (p_2 p_0)$, , and that all 1-cycles are linear combinations of ~~this~~ this one (i.e. multiples of this one). But to prove it, first look at $C_1(K)$:

$$\begin{aligned} C_1(K) &= \text{gen} \{ (p_0 p_1), (p_1 p_2), (p_2 p_0) \} \\ &= \left\{ \cancel{n_2} n_2 (p_0 p_1) + n_0 (p_1 p_2) + n_1 (p_2 p_0) \mid n_0, n_1, n_2 \in \mathbb{Z} \right\} \cong \mathbb{Z}^3. \end{aligned}$$

Now demand that an arbitrary element of $C_1(K)$ be a cycle:

$$\begin{aligned} 0 &= \partial [n_2 (p_0 p_1) + n_0 (p_1 p_2) + n_1 (p_2 p_0)] \\ &= (n_1 - n_2)(p_0) + (n_2 - n_0)(p_1) + (n_0 - n_1)(p_2) \end{aligned}$$

$$= 0 \quad \text{iff} \quad \left. \begin{array}{l} n_1 = n_2 \\ n_2 = n_0 \\ n_0 = n_1 \end{array} \right\} \quad \text{i.e.} \quad n_0 = n_1 = n_2.$$

So,

$$\mathbb{Z}_1(K) = \text{gen} \{ (p_0 p_1) + (p_1 p_2) + (p_2 p_0) \} \cong \mathbb{Z},$$

as expected, and

$$H_1(K) = \mathbb{Z}.$$

Note the difference between \mathcal{I} and Δ : both are 1-dimensional, but

$$H_1(\mathcal{I}) = \{0\}$$

$$H_1(\Delta) = \mathbb{Z}.$$

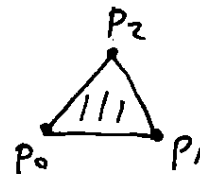
The \mathbb{Z} in the second case reflects the existence of one cycle that is not a boundary. It indicates the existence of a "hole" in the space.

Note also that Δ is homeo. to S^1 , so we now have the homology groups for S^1 :

$$H_0(S^1) = \mathbb{Z},$$

$$H_1(S^1) = \mathbb{Z}.$$

[4.] A triangle (the full 2D object):



$$K = \{ (p_0 p_1 p_2), (p_0 p_1), (p_1 p_2), (p_2 p_0), (p_0), (p_1), (p_2) \}.$$

Same K as in last example, except $(p_0 p_1 p_2)$ now included.

So, $\dim K = n = 2$.

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Again $H_0(K) = \mathbb{Z}$. ($r=0$). Next, $r=2$.

$H_2(K) = Z_2(K)$. The only 2-chain is a multiple of $(P_0P_1P_2)$, but $\partial(P_0P_1P_2) = (P_0P_1) + (P_1P_2) + (P_2P_0) \neq 0$, so the only 2-cycle is 0 and $H_2(K) = \{0\}$.

Now $r=1$.


$$H_1(K) = \frac{Z_1(K)}{B_1(K)}$$

First look at $Z_1(K)$. It is the same as in the previous example ($C_1(K)$ is the same, too), since $C_1(K)$ is generated by same generators. But $B_1(K)$ is different, since now there are 2-chains whose bdy we can take. In fact, from above we have

$$B_1(K) = \text{gen} \{ (P_0P_1) + (P_1P_2) + (P_2P_0) \} = Z_1(K) \cong \mathbb{Z}$$

$$\text{So, } H_1(K) = \frac{Z_1(K)}{B_1(K)} \cong \frac{\mathbb{Z}}{\mathbb{Z}} \cong \{0\}.$$

The reason $H_1(K)$ changed from \mathbb{Z} to $\{0\}$ on going from example [3] to [4] is because the hole disappeared.

[5]. A tetrahedron (i.e., surface of a tetrahedron), a 2-dim simplicial complex, B_2 , homeo to 2-sphere S^2 .

We'll use some intuition and general principles.

First, $r=0$. Easy, $H_0(K) = \mathbb{Z}$.

Next, $r=2$. $H_2(K) = Z_2(K)$. Are there any 2-cycles?

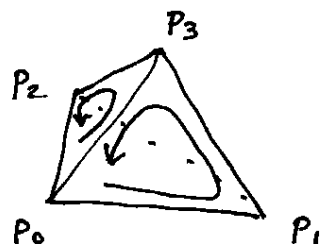
Intuitively, a 2-cycle is a "closed surface", and the sphere S^2 is certainly one of these (hence also the tetrahedron).

Thus we expect that $\partial(S^2) = 0$, where (S^2) stands for the 2-chain that is the whole surface of the tetrahedron, actually the sum of 4 2-simplexes, ^{2/5/04}

$$(S^2) = (P_0 P_3 P_2) + (P_0 P_1 P_3) + (P_2 P_3 P_1) + (P_0 P_2 P_1).$$

Here each of the faces has been oriented by the "right hand rule" for an "outward pointing normal". You can check directly that $\partial(S^2) = 0$.

Another way is to note that the common edges of any 2 adjacent triangles ^(faces) is oppositely oriented by those triangles,



So the common edge cancels when we take the boundary. In fact, the most general 2-chain is a linear combination of these 4 triangles, and the only way that its boundary can vanish is if the coefficients of all adjacent triangles (hence all triangles) are equal. Thus, every 2-cycle is a multiple of (S^2) itself, confirming our intuition. Hence $Z_2(K) = \mathbb{Z} = H_2(K)$.

Finally, case $r=1$. Question: Are there any 1-cycles that are not boundaries? If you liberate yourself from the triangulation, the answer is no (you can see), because any closed loop on S^2 is clearly a boundary:



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Alternatively, note that all 1-cycles on the sphere S^2 can be contracted to a point (the zero 1-chain), so therefore are boundaries.

Hence $Z_1(K) = B_1(K)$ and $H_1(K) = \{0\}$.

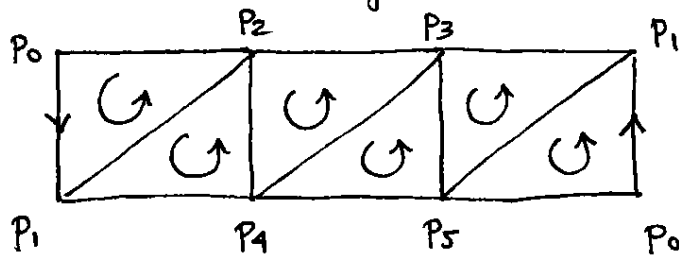
Notice two examples of spheres so far, dim 1 and 2:

	$H_0(K)$	$H_1(K)$	$H_2(K)$
S^1	\mathbb{Z}	\mathbb{Z}	
S^2	\mathbb{Z}	$\{0\}$	\mathbb{Z} .

↙ means S^2 is "closed".

Can generalize, show that for S^n , $K_0(S^n) = \mathbb{Z}_0 = K_n(S^n)$, and $K_r(S^n) = \{0\}$ for $0 < r < n$. Intuitively, the reason for the ^{triviality} ~~vanishing~~ of $H_r(S^n)$ for $1 \leq r \leq n-1$ is the same as that for $H_1(S^2)$: The r -cycles on the face of S^n are contractible to a point.

⑥ The Möbius strip. Regard it as a square with opposite sides identified (in reverse order), and triangulate it as we did the cylinder.

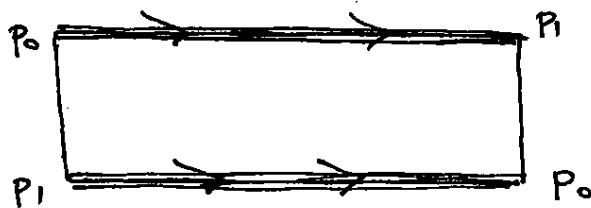


orient the triangles as shown.

Case $r=0$ easy (as always), $H_0(K) = \mathbb{Z}$.

Now case $r=2$. For sphere we found $\partial(S^2) = 0$. What is $\partial(M\ddot{o})$, where $(M\ddot{o})$ is the whole Möbius strip, the sum of all 6 ~~rectangles~~ triangles (2-simplexes) as shown? Unlike the sphere, $M\ddot{o}$ has an edge (only one, actually), so that if a triangle has an edge that is part of the edge of $M\ddot{o}$, then that edge (which appears when you apply ∂ to $(M\ddot{o})$) can never be cancelled by an adjacent triangle. (The same would be true for any space with a boundary, e.g., the cylinder). In fact, all triangles in $(M\ddot{o})$ have at least one edge that is a part of the edge of $(M\ddot{o})$ (there are no purely "internal" triangles).

A guess for what the edge of $M\ddot{o}$ would be is



Heavy lines are the path.

i.e., $P_0 \rightarrow P_1 \xrightarrow{\text{back}} P_0$ again along the obvious "edge" you would see if you constructed a Möbius strip out of paper. But this is not what you get when you take $\partial(M\ddot{o})$ where

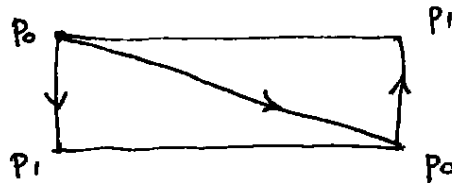
$$(M\ddot{o}) = \sum (\text{six 2-cycles})_{\text{shown}}$$

i.e. the six triangles with the orientation shown. Instead you get

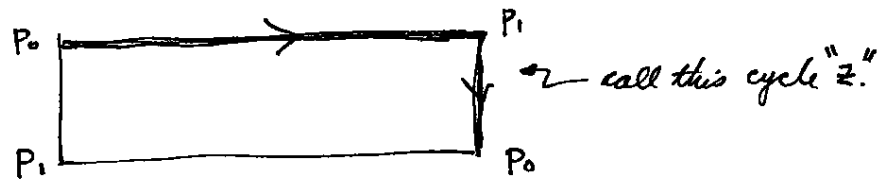
$$\partial(M\ddot{o}) = \text{[Diagram of Möbius strip with boundary arrows]} = \text{[Diagram of Möbius strip with boundary arrows]} + 2 \left(\text{[Diagram of a triangle with vertices } P_0 \text{ and } P_1 \text{]} \right)$$

So there are 2 reasons why $\partial(M\ddot{o})$ does not vanish: $M\ddot{o}$ has an edge, so abutting triangles have edges that can't be cancelled, and secondly because even the internal edges cannot be all cancelled because you can't orient the triangles "coherently" (so that all internal edges cancel due to opposite orientations of adjacent triangles). The latter effect is due to the non-orientability of the Möbius strip. In fact, it's easy to see that no linear combination of any set of triangles with any orientation can give a 2-cycle. There are no 2-cycles, $Z_2(K) = \{0\}$ and $H_2(K) = \{0\}$.

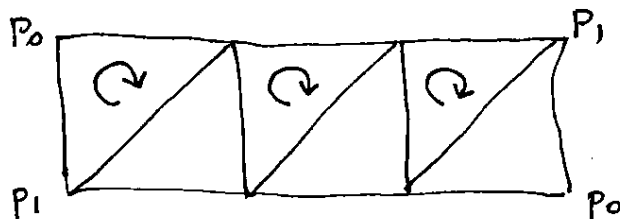
Now case $r=1$. $H_1(K)$ of course contains the element 0, the equivalence class of 1-cycles that are also boundaries, for example the boundary of any triangle. Are there one-cycles that are not boundaries? Inspection suggests one candidate,



(the diagonal line). This is drawn without regard to the triangulation, but by contin. deformation we can make it run along the available 1-simplexes (edges):



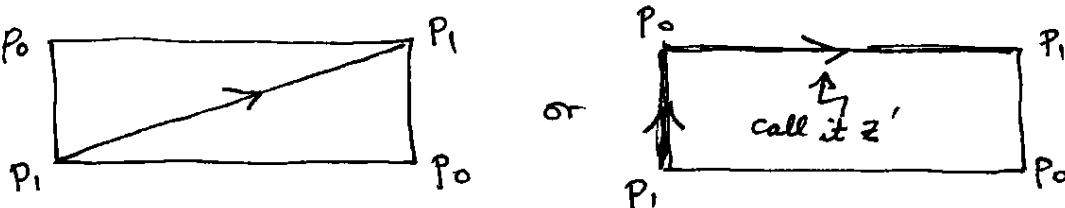
This cycle is not a boundary. If it were, it would have to be the boundary of a sum of triangles that at a minimum would include those with edges along the top:



But these have uncancelled internal edges, which you can't get rid of unless you add more triangles to the mix, whereupon you get the whole (Mö), whose boundary is not \mathbb{Z} . So $\mathbb{Z} \neq 0$ is a cycle that is not a boundary.

Are there any other, independent ones, that is, 1-cycles ~~and~~ that are not boundaries and are not homologous to some multiple of \mathbb{Z} ?

We might try



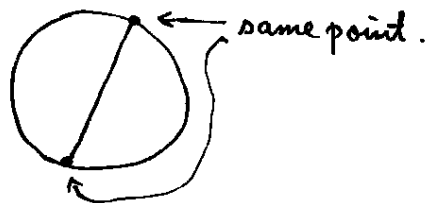
But z' can be deformed into z , so z and z' should be homologous. Indeed, if we calculate we find

$$z - z' = \partial(Mö)$$

There is only one independent 1-cycle ^{z} that is not a boundary, and $[z]$ generates $H_1(K)$:

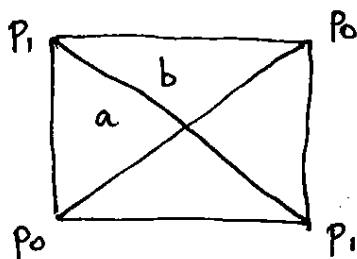
$$H_1(K) = \{ n[z] \mid n \in \mathbb{Z} \} = \mathbb{Z}$$

7 Now $\mathbb{R}P^2$, which can be realized as disk with opposite points on circumference identified:



First we have to triangulate.

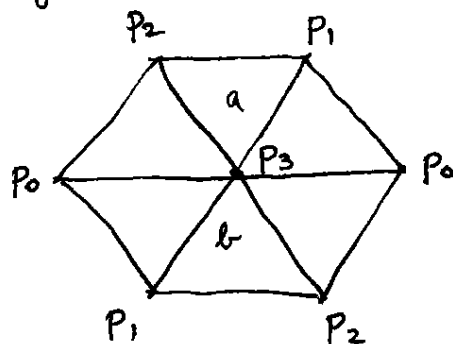
A square won't work:



because triangles a, b are identical in K .

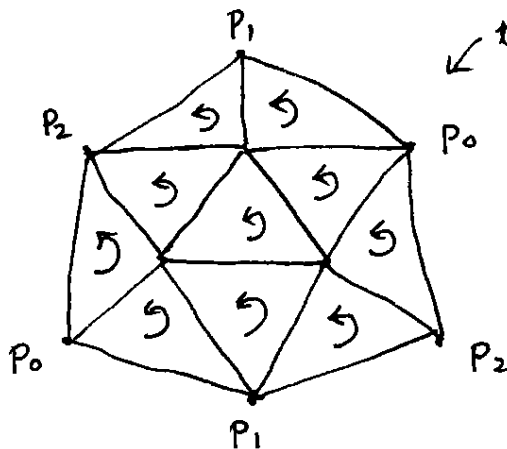
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So try a hexagon:



Wont work,
again because triangles a, b
are identical.

So use one internal triangle,



← this works.

and orient the triangles as
shown.

- 10 triangles (faces)
- 15 edges
- 6 vertices.

$$\left. \begin{aligned} C_2(k) &\cong \mathbb{Z}^{10} \\ C_1(k) &\cong \mathbb{Z}^{15} \\ C_0(k) &\cong \mathbb{Z}^6 \end{aligned} \right\} \text{ pretty large dimensions}$$

$r=0$ trivial, $H_0(k) = \mathbb{Z}$.

$r=2$: As always, $H_n(k) = \mathbb{Z}_n(k)$ (here $n=2$). Are there any 2-cycles (closed surfaces)? Unlike the Möbius strip, $\mathbb{R}P^2$ has no edge, so there are no triangles with edges that are not shared with another triangle. In that sense $\mathbb{R}P^2$ is a "closed" surface. But, the orientations of the triangles are not coordinated, i.e., not all common edges have opposite orientation. To put it another way: if we want to construct a 2-cycle, it must be a linear comb. of the triangles shown. If it contains any one of the triangles shown, then it must include its ~~neighbor~~ neighbors in order to cancel the internal edges. So it must include the whole set of 10 triangles. Call this $(\mathbb{R}P^2) = \sum (\text{ten triangles w. orientation shown})$.

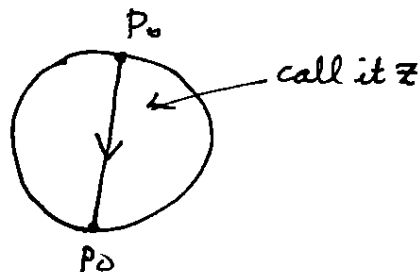
But

$$\partial(\mathbb{R}P^2) = 2[(p_0 p_1) + (p_1 p_2) + (p_2 p_0)] \neq 0.$$

Therefore there are no 2-cycles (apart from 0), and $H_2(K) = \{0\}$.

Now $r=1$: We need to find 1-cycles that are not boundaries.

An obvious choice is



which can be distorted to run along the triangulation, in fact, we can identify z with $(p_0 p_1) + (p_1 p_2) + (p_2 p_0)$. This is not a boundary (use same logic as on Möbius strip).

Unlike the case of the Möbius strip however, $2z = \partial(\mathbb{R}P^2)$ is a boundary! So $[z]$ is a generator of $H_1(K)$, but it obeys the rule $2[z] = 0$.

By experimenting, we conclude that there are no other 1-cycles, indep. of z and not a boundary. Thus,

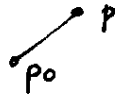
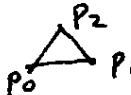
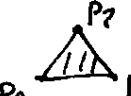
$$H_1(K) = \text{gen}\{[z]\} \cong \mathbb{Z}_2.$$

This is our first example of one of the \mathbb{Z}_k groups in a homology group. This is sometimes described as being due to the "twisting" of the space.

Skip other examples in book (torus, genus g surface Σ_g , Klein bottle).

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A table.

	K	$H_0(K)$	$H_1(K)$	$H_2(K)$
1.	$\bullet p_0$	\mathbb{Z}		
2.	$\bullet p_0 \quad \bullet p_1$	\mathbb{Z}^2		
3.		\mathbb{Z}	$\{0\}$ ← no 1-cycles	
4.		\mathbb{Z}	\mathbb{Z}	
5.		\mathbb{Z}	$\{0\}$	$\{0\}$ ← no 2-cycles
5 1/2.	Tetrahedron $\cong S^2$	\mathbb{Z}	$\{0\}$	\mathbb{Z} ← \exists one 2-cycle, the surface S^2 itself.
6.	Möbius	\mathbb{Z}	\mathbb{Z}	$\{0\}$
7.	$\mathbb{R}P^2$	\mathbb{Z}	\mathbb{Z}_2	$\{0\}$
8.	Klein bottle	\mathbb{Z}	$\mathbb{Z} \times \mathbb{Z}_2$	$\{0\}$
9.	Torus T^2	\mathbb{Z}	$\mathbb{Z} \times \mathbb{Z}$	\mathbb{Z}

The example $H_1(K)$ for the Klein bottle shows the general form of homology groups,

$$H_r(K) \cong \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{f \text{ factors}} \times \underbrace{\mathbb{Z}_{k_1} \times \dots \times \mathbb{Z}_{k_p}}_{\text{this part called the torsion group.}}$$

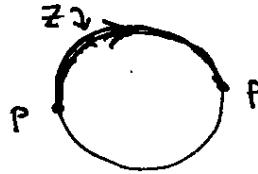
↑
this is the free subgroup.

↘ this part called the torsion group.

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Now some remarks on what can happen when we allow real coefficients when forming chains. To distinguish the cases, we'll write $H_r(K, \mathbb{Z})$ for the homology group in which chains are restricted to integer coefficients, and $H_r(K, \mathbb{R})$ for real coefficients. Also may consider $H_r(K, \mathbb{Z}_2)$ (these are the most popular choices).

As an illustration, consider the 1-cycle z in $\mathbb{R}P^2$,



Under \mathbb{Z} coefficients, z is not a boundary, but $2z = \partial(\mathbb{R}P^2)$ is. This $[z]$ generates the torsion subgroup \mathbb{Z}_2 of $H_1(K, \mathbb{Z})$. But if we allow real coefficients, then z is a boundary, i.e., of the 2-chain $\frac{1}{2}(\mathbb{R}P^2)$. So with \mathbb{R} coefficients, there are no 1-cycles that are not boundaries, and

$$H_1(\mathbb{R}P^2, \mathbb{R}) = \{0\} = \mathbb{R}^0.$$

More generally, when we compute quotient groups,

$$H_r(K, \mathbb{R}) = \frac{Z_r(K, \mathbb{R})}{B_r(K, \mathbb{R})}$$

it is always of the form $\frac{\mathbb{R}^n}{\mathbb{R}^m}$, $n \geq m$, which $= \mathbb{R}^{n-m}$. There is

no torsion subgroup, but the free part has the same dimensionality as in the case $H_r(K, \mathbb{Z})$. That is,

$$\begin{aligned} \text{If } H_r(K, \mathbb{Z}) &= \mathbb{Z}^f \times \mathbb{Z}_{k_1} \times \dots \times \mathbb{Z}_{k_p} \\ \text{then } H_r(K, \mathbb{R}) &= \mathbb{R}^f \quad (\text{free part only}). \end{aligned}$$

The dimension of the free part of $H_r(K, \mathbb{Z})$, which is the dimension of $H_r(K, \mathbb{R})$, is called the r -th Betti number of K , denoted

$$b_r(K) = \dim H_r(K, \mathbb{R}).$$

There is an interesting relation between the Euler characteristic and the Betti numbers.

The usual Euler characteristic on a 2D surface S (sphere, torus) is

$$\chi(S) = f - e + v,$$

$$f = \# \text{ faces} \quad (\# \text{ 2-simplexes in } K) \quad \Rightarrow$$

$$e = \# \text{ edges} \quad (\# \text{ 1-simplexes in } K)$$

$$v = \# \text{ vertices} \quad (\# \text{ 0-simplexes in } K).$$

On the sphere S^2 , $\chi = 2$, independent of the triangulation. The obvious generalization of this is to replace f, e, v etc. by the dimension of the r -th chain group $C_r(K, \mathbb{R})$, which is just the number of independent r -chains in K . So, define

$$\chi(K) = \sum_{r=0}^n (-1)^r \dim C_r(K, \mathbb{R}). \quad (n = \dim K).$$

(These dimensions are not topological invariants.)

Now, ∂_r is a map: $C_r(K) \rightarrow C_{r-1}(K)$, so

(under \mathbb{R} , not \mathbb{Z} , so ∂ is a linear map),

$$\dim \ker \partial_r + \dim \text{im } \partial_r = \dim C_r(K, \mathbb{R}).$$

$$\text{or} \quad \dim Z_r(K) + \dim B_{r-1}(K) = \dim C_r(K)$$

(over \mathbb{R}
understood
now.)

In the case $r=0$, there is no $B_{-1}(K)$, but $Z_0 = C_0$ so we may interpret $\dim B_{-1}(K)$ as zero.

So,

$$\chi(k) = \sum_{r=0}^n (-1)^r \left[\dim Z_r(k) + \dim B_{r-1}(k) \right].$$

$$\text{2nd sum} = \sum_{r=0}^n (-1)^r \dim B_{r-1}(k)$$

$$= \sum_{r=1}^n (-1)^r \dim B_{r-1}(k) \quad (\text{drop } B_{-1} \text{ term})$$

$$= - \sum_{r=0}^{n-1} (-1)^r \dim B_r(k) \quad (\text{shift index})$$

$$= - \sum_{r=0}^n (-1)^r \dim B_r(k) \quad (\text{add } r=n \text{ term, } \overset{\dim}{B_n}(k)=0).$$

So whole sum becomes,

$$\chi(k) = \sum_{r=0}^n (-1)^r \left[\dim Z_r(k) - \dim B_r(k) \right]$$

$$= \sum_{r=0}^n (-1)^r \dim H_r(k) = \sum_{r=0}^n (-1)^r b_r(k).$$

This is the Euler-Poincaré thm, relating the Euler-characteristic to the Betti numbers, manifest topological invariants.