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Now we turn to the machinery needed to formalize the idea of a polyhedron.

<u>Def.</u> A <u>r-simplex</u> is a region of $\mathbb{R}^m$ $(m \geq r)$ specified by $r+1$ distinct points, $p_0, p_1, \ldots, p_r$, denoted $\sigma = \langle p_0 p_1 \cdots p_r \rangle$, and defined by

$$\sigma = \langle p_0 p_1 \cdots p_r \rangle = \left\{ \sum_{i=0}^{r} c_i p_i \;\middle|\; \sum_{i=1}^{r} c_i = 1, \quad c_i \geq 0 \right\}.$$

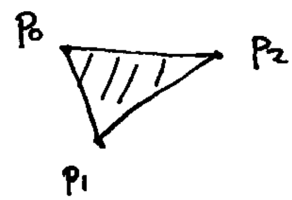
Coefficients $\{c_i\}$ are <u>barycentric coordinates</u>. The <u>order</u> of the points in $\langle p_0 \cdots p_r \rangle$ is unimportant. (any permutation represents the same simplex.)

<u>Examples:</u> $\sigma_0 = \langle p_0 \rangle$ = a point

$\sigma_1 = \langle p_0 p_1 \rangle$ = an edge

$x = c_0 p_0 + c_1 p_1$
$= c_0 p_0 + (1 - c_0) p_1$
$0 \leq c_0 \leq 1.$

$\sigma_2 = \langle p_0 p_1 p_2 \rangle$ = a face (triangle)



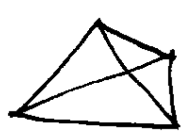
<u>Def.</u> Let $\sigma_r = \langle p_0 \ldots p_r \rangle$ = a simplex

$\sigma_q = \langle p_{i_0} \cdots p_{i_q} \rangle$ = a q-face of σ_r, $(q \leq r)$

where $\{p_{i_0}, \cdots p_{i_q}\}$ is a subset of $\{p_0, \ldots, p_r\}$.

Denote $\sigma_q \leq \sigma_r$.

<u>Example:</u>



σ_3 = tetrahedron = 3-simplex

4 2-faces
6 1-faces
4 0-faces

$\#$ q-faces $= \binom{r+1}{q+1}$

<u>Def.</u> A <u>simplicial complex</u> is a set $K = \{\sigma_i\}$ of simplexes such that:

1) If $\sigma \in K$ then K also contains all the faces of σ
2) If $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \phi$ or σ and σ' intersect in a common face.

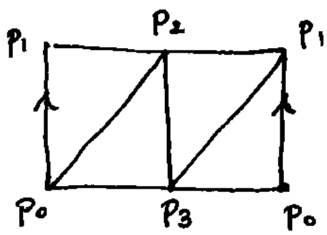
Property 2 means that simplicial complexes are "nicely fitted together"

<u>Def.</u> A <u>polyhedron</u> $|K|$ is the set

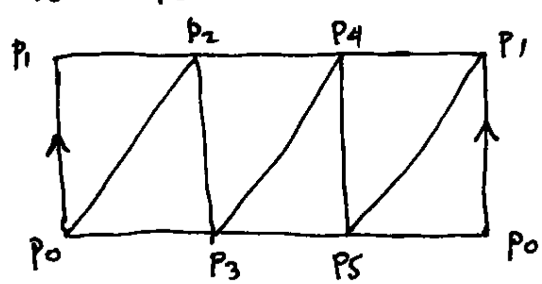
$$|K| = \bigcup_{\sigma_i \in K} \sigma_i .$$

<u>Def.</u> A topological space X is <u>triangulable</u> if $\exists$ a polyhedron $|K|$ and homeomorphism $f: |K| \to X$.

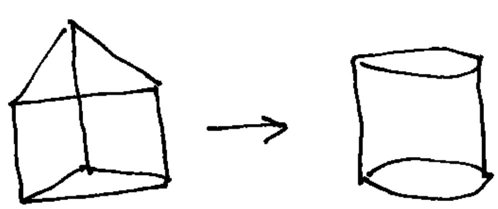
<u>Example:</u> A cylinder (square with 2 sides identified, ).

This 

doesn't work, because $|K|$ is not homeomorphic to the cylinder (it is two rectangles superimposed, i.e., one rectangle).

Instead, use 

6 triangles.

$f:$ 

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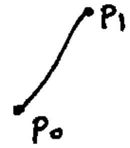
Now oriented simplexes. Look at e.g. 1-simplex.

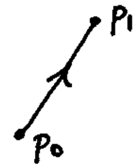
$$\langle p_0 p_1 \rangle = $$



$$= \langle p_1 p_0 \rangle \quad \text{unordered.}$$

Change of notation, write $(p_0 p_1)$ for ordered simplex,

$$(p_0 p_1) = $$



$$= -(p_1 p_0).$$

For 2-simplexes, have 3 points, $(p_0 p_1 p_2)$. Declare that this changes sign if points subjected to an odd permutation. Generally,

$$(p_{i_0} p_{i_1} \cdots p_{i_n}) = \pm (p_0, \ldots, p_n)$$

$$\text{where } \pm = \text{sign of permutation } \begin{pmatrix} 0 & 1 & \cdots & n \\ i_0 & i_1 & \cdots & i_n \end{pmatrix}.$$

Special case of 0-dimension. $\langle p_0 \rangle = \bullet$ a point.

(p_0) = "oriented" point.

As discussed previously, motivation for oriented simplexes is use in integrals, e.g., Stokes' thm. Here is what we mean by a "0-dimensional integral":

$$\int_{(p_0)} f = f(p_0) \qquad \text{where } p_0 \in M, \quad f : M \to \mathbb{R} \ \text{(say)}.$$

Thus,

$$\int_{-(p_0)} f = -f(p_0), \text{ etc.}$$

Thus linear combinations of oriented simplexes become meaningful (as things you integrate over).

Above we defined a simplicial complex K as a collection of unoriented simplexes. Now we modify the definition in obvious ways to talk about an <u>oriented simplicial complex</u> K: it is a set of <u>oriented simplexes</u> $\{\sigma_i\}$. (Use the same symbols.)

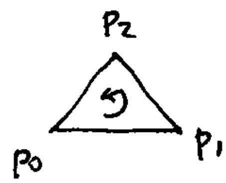
Let $K =$ an oriented simplicial complex $= \{\sigma_i\}$. Define:

<u>Def</u>: The <u>r-th chain group</u> $C_r(K)$ is the free Abelian group generated by the r-dimensional, oriented simplexes in K, that is, it is the set of formal linear combinations,

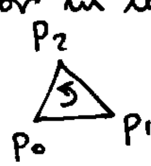
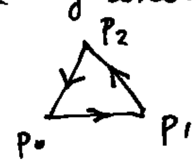
$$c = \sum_{i}^{I_r} n_i \, \sigma_{ri} , \qquad n_i \in \mathbb{Z} , \qquad c \in C_r(K).$$

where $\{\sigma_{ri}\}$ is the set of r-simplexes in K, whose number is I_r. c is called an <u>r-chain</u>. Note, $C_r(K) \cong \mathbb{Z}^{I_r}$.

Now we motivate the definition of the <u>boundary operator</u>. Start with an oriented 2-simplex $(p_0 p_1 p_2)$



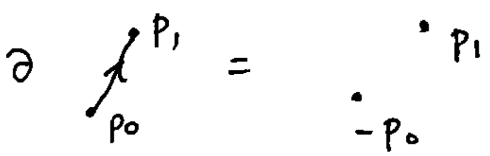
The arrow is a convenient way of specifying this orientation, since any cyclic permutation of $(p_0 p_1 p_2)$ is an even permutation. (The opposite direction would reverse the sign.) We define the boundary operator in this example by writing

∂  $=$  or

$$\partial \, (p_0 p_1 p_2) = (p_0 p_1) + (p_1 p_2) + (p_2 p_0).$$

Notice that ∂ acting on a 2-simplex is not a simplex, but rather a linear combination of simplexes (a chain). Another example, the boundary of a 1-simplex:

$$\partial\,(p_0 p_1) = (p_1) - (p_0)$$



In general, we define $\partial_r : C_r(K) \to C_{r-1}(K)$ by giving its action on an r-simplex and then extending in the obvious way to linear combinations. If $\sigma_r = (p_0, \dots, p_r)$ is an oriented r-simplex, then we define

hat means omit this point.

$$\partial_r \sigma_r = \sum_{i=0}^{r} (-1)^i \,(p_0 \dots \hat{p}_i \dots p_r)$$

E.g., $\partial_2\,(p_0 p_1 p_2) = (p_1 p_2) - (p_0 p_2) + (p_0 p_1)$, same answer as before since $-(p_0 p_2) = +(p_2 p_0)$.

One more note about defn of bdry operator. In special case $r=0$, we define

$$\partial_0\,(p_0) = 0,$$

since there are no (-1)-chains.

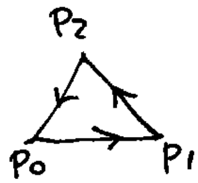
Also note, $\partial_r : C_r(K) \to C_{r-1}(K)$ is a group homomorphism (it commutes with $+$).

There is really one bdry operator for each dimension. Each maps $C_r(K)$ into $C_{r-1}(K)$. So we have a sequence of maps, ($n = \dim K$) 2/3/04

$$C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} \{0\}.$$

Def: An r-chain $c \in C_r(K)$ such that $\partial c = 0$ is called a cycle (or r-cycle). A cycle is a chain without a boundary.

Example:



$$c = (p_0 p_1) + (p_1 p_2) + (p_2 p_0)$$

$$\partial c = (p_1) - (p_0) + (p_2) - (p_1) + (p_0) - (p_2) = 0.$$

c is a cycle (1-cycle).

another example: Any 0-chain is a cycle, since $\partial(p_0) = 0$.

Def: The set

$$Z_r(K) = \{ c \in C_r(K) \mid \partial_r c = 0 \} = \ker \partial_r$$

is the r-th cycle group. It is obviously a subgroup of $C_r(K)$, which means (see notes with HW 2) that it is a sublattice of $C_r(K)$ and that it (like $C_r(K)$) is a free, finitely generated Abelian group.

Special case: ($r=0$) $Z_0(K) = C_0(K)$ (all 0-chains are cycles).

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__Def__: The set

$$B_r(K) = \{ b \in C_r(K) \mid b = \partial_{r+1} c, \text{ some } c \in C_{r+1}(K) \} = \operatorname{im} \partial_{r+1}$$

is the _r-th boundary group_. Elements of $B_r(K)$ are called _r-boundaries_. $B_r(K)$ is obviously a subgroup of $C_r(K)$, hence it is a free, finitely generated Abelian group.

Special _case_: For $r = n$, since there are no $(n+1)$-simplexes, we define $B_n(K) = \text{"} \operatorname{im} \partial_{n+1} \text{"} = \{0\}$.

__Thm__: $\partial_r \partial_{r+1} = 0$. The boundary of a boundary vanishes.

Proof: It suffices to consider an $(r+1)$-simplex (a basis vector in $C_{r+1}(K)$), $(p_0, \ldots, p_{r+1})$:

$$\partial_{r+1}(p_0, \ldots, p_{r+1}) = \sum_{i=0}^{r+1} (-1)^i (p_0 \ldots \hat{p}_i \ldots p_{r+1})$$

$$\partial_r \partial_{r+1}(p_0, \ldots, p_{r+1}) = \sum_{i=0}^{r+1} (-1)^i \Big[\sum_{j=0}^{i-1} (-1)^j (p_0 \ldots \hat{p}_j \ldots \hat{p}_i \ldots p_{r+1}) + \sum_{j=i+1}^{r+1} (-1)^{j+1} (p_0 \ldots \hat{p}_i \ldots \hat{p}_j \ldots p_{r+1}) \Big] = 2 \text{ terms}$$

~~Swap i,j in 2nd term, it becomes~~

$$\cancel{- \sum_{i=0}^{r+1} (-1)^i \sum_{j=}}$$

1st term $= \sum_{\cancel{i<j}\; i>j} (-1)^{i+j} \, (p_0 \ldots \hat{p}_j \ldots \hat{p}_i \ldots p_{r+1})$

2nd term $= - \sum_{\cancel{i>j}\; i<j} (-1)^{i+j} \, (p_0 \ldots \hat{p}_i \ldots \hat{p}_j \ldots p_{r+1}) =$ $-$ 1st term by swapping i, j.

QED.

Immediate corollary: $B_r(K) \subseteq Z_r(K)$.

If $b \in B_r(K)$, then $b = \partial c$, some $c \in C_{r+1}(K)$.

Hence $\partial b = \partial\partial c = 0$ hence $b \in Z_r(K)$.

Altogether, $\boxed{B_r(K) \subseteq Z_r(K) \subseteq C_r(K)}$

Actually $\subseteq$ means "subgroup". All 3 groups are free, finitely generated Abelian group.

Finally, we define

Note, for $r > \dim K$, $H_r(K)$ is understood to be $\{0\}$ (the trivial group).

$$H_r(K) = \frac{Z_r(K)}{B_r(K)} = r\text{-th homology group.}$$

$H_r(K)$ is a topological invariant, that is, if $|K|$ and $|K'|$ are homeomorphic, then $H_r(K) = H_r(K')$. If you have a topological space X homeomorphic to $|K|$, then $H_r(X)$ is regarded as the same as $H_r(K)$.

Note, if $h \in H_r(K)$, then h is an equivalence class of cycles whose difference is a boundary,

$$h = [c], \quad c \in Z_r(K),$$
$$c \sim c' \text{ if } c - c' \in B_r(K).$$

Note that the zero element in $H_r(K)$ is the equivalence class of boundaries,

$$0 \in H_r(K) \text{ means}$$
$$0 = B_r(K).$$

Some general features of homology groups. First take case $r=0$.
As noted above, all 0-simplexes (p) are automatically cycles.

Note that the boundary of a 1-simplex is always the difference between the endpoints,

$$\partial\left(\overline{pq}\right) = (q) - (p),$$

which shows that any two 0-simplexes (p) and (q) are homologous if the points p and q can be connected by a curve.

$$(p) = (q) - \left[(q) - (p)\right].$$

In fact this is iff. So if a manifold M consists of N disconnected pieces,

Then all 0-simplexes (p) where p belongs to one piece are homologous to all other simplexes (q) where q belongs to the same piece. So the equivalence classes of cycles are generated by $[(p_1)], \dots [(p_N)]$ where $p_1, \dots, p_N$ are taken from each piece. Thus, a general element of $H_0(M)$ has the form,

$$h \in H_0(M),$$
$$h = \sum_{i=1}^{N} n_i \left[(p_i)\right],$$

and $\boxed{H_0(M) = \mathbb{Z}^N, \quad N = \#\text{ of disconnected pieces of } M.}$

In particular, for a connected manifold, $H_0(M) = \mathbb{Z}$.