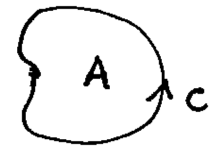
1/29/04

Begin today with motivation for <u>homology theory</u>, ch. 3 of Nakahara. Nakahara's presentation starts out rather technically, so motivate some first.

Consider Stokes' Theorem in 3D ($\mathbb{R}^3$):

$$\int_C \vec{F}\cdot d\vec{x} = \int_A \nabla\times\vec{F}\cdot d\vec{a}$$

Both C and A have an <u>orientation</u>.




Curve C is <u>boundary</u> of area A.

C, A examples of <u>chains</u> (defined later).

$$\left.\begin{array}{l} C = \text{1-chain} \\ A = \text{2-chain} \end{array}\right\} \text{things you integrate over.}$$

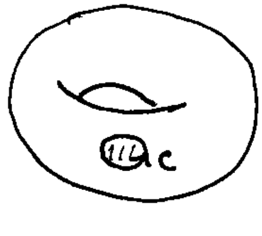
Write,

$$C = \partial A$$

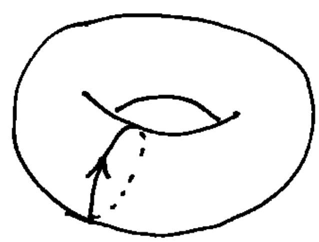
$\uparrow$ <u>boundary operator</u>

C is a <u>closed</u> <u>curve</u>, called a <u>cycle</u> (or 1-cycle).

On $\mathbb{R}^3$, every closed curve (1-cycle) is the boundary of some 2D region A. But on other spaces this is not true. The 2-torus, for example

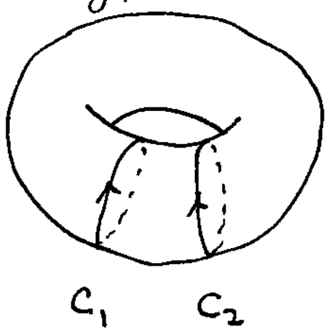


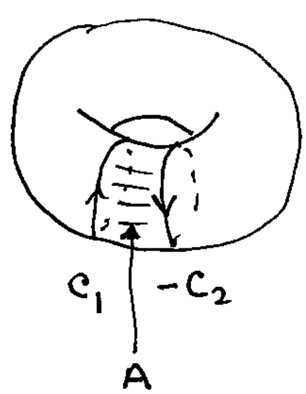

$C = \partial A$



$C \neq \partial A$

This fact conveys topological information about the manifold. More precisely, introduce equivalence classes of cycles.




We say, $C_1 \sim C_2$ if

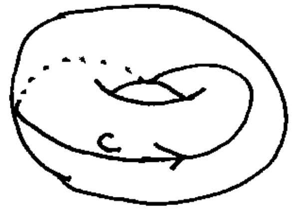
$$C_1 - C_2 = \partial A.$$

some A.

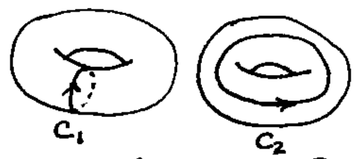
We say, $C_1$ and $C_2$ are <u>homologous</u>.

On $\mathbb{R}^3$, every 1-cycle is a boundary, $C_1 = \partial A_1$, $C_2 = \partial A_2$, so their difference is a boundary, too, $C_1 - C_2 = \partial(A_1 - A_2)$, and all 1-cycles belong to the same equivalence class. On $\mathbb{R}^3$, there is only one equivalence class. $[c]$, any 1-cycle $c$, in particular, $c = 0$ (the curve that doesn't go anywhere).

But on the 2-torus, two 1-cycles are equivalent iff their "winding numbers" $n_1, n_2$ are the same.



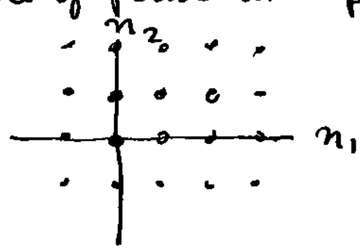

$c$ has winding numbers $(1, 1)$.




$c = c_1 + c_2$.

Thus the equivalence classes of 1-cycles on $T^2$ are numbered by 2 integers $(n_1, n_2)$. The space of equivalence classes is $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$.

Can think of this as lattice of points in a plane:




Can also think of it as an <u>Abelian</u> <u>group</u>, with composition law

$$(n_1, n_2)\,(m_1, m_2) = (n_1 + m_1, n_2 + m_2).$$

(vector addition of lattice vectors). This is expressed by saying,

$$H_1(T^2) = \mathbb{Z}^2$$

$$H_1(\mathbb{R}) = \{0\}.$$

$H_1$ (manifold) = 1st <u>homology</u> <u>group</u> of the manifold.

It is a <u>topological</u> <u>invariant</u>.

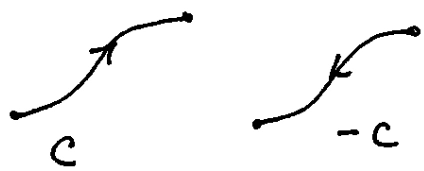
What does it mean to use expressions like $-C$, $C_1+C_2$, $C_1-C_2$, etc?

Let $C$ = any oriented curve on a manifold $M$ (not necessarily a cycle).


Formally, this is a map $C: [0,1] \to M : t \mapsto x(t)$. It is something you integrate over,

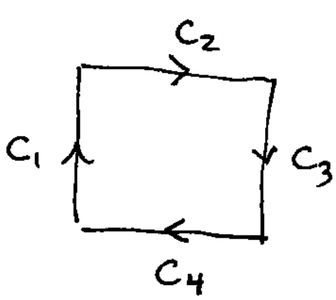
$$\int_C \alpha$$

where $\alpha$ = a differential 1-form (like $\vec{F}\cdot d\vec{x}$ on $\mathbb{R}^3$). Let $-C$ be the same curve traversed in the opposite direction,




So that $$\int_{-C} \alpha = -\int_C \alpha.$$

If you have several segments of a curve, (not necessarily concatenated) define their sum by sums of integrals:




$$\int_{C_1+C_2+C_3+C_4} \alpha = \int_{C_1} \alpha + \int_{C_2} \alpha + \int_{C_3} \alpha + \int_{C_4} \alpha.$$

Thus we can define "linear combinations" of curves with integer coefficients, and

$$\int_{\sum_i n_i c_i} \alpha = \sum_i n_i \int_{c_i} \alpha, \qquad n_i \in \mathbb{Z}.$$

These linear combinations of 1-dimensional, oriented curves with integer coefficients are called <u>1-chains</u>. They are things you integrate 1-forms over.

Thus a 1-chain is a formal linear combination of oriented 1-curves with <u>integer</u> <u>coefficients</u>.

[Actually, in homotopy theory, one can choose the coefficients to be things other than integers. The favorite choices are $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{Z}_2$ (the spaces from which the coefficients are chosen). For now we use only $\mathbb{Z}$, but later we'll return to other types of coefficients.]

There is an huge infinite number of possible oriented curves on a given manifold. In our approach to homotopy theory, we will reduce this to a finite number by using a <u>triangulation</u> of a manifold (this is a homeomorphism between a <u>polyhedron</u> in $\mathbb{R}^n$ and the manifold M we wish to study).

The set of 1-chains that can be formed out of a finite # of N distinct oriented curves $\{C_1, \ldots, C_N\}$ is

$$\sum_{i=1}^{N} n_i C_i \ , \quad n_i \in \mathbb{Z},$$

it is the space $\mathbb{Z}^N = \mathbb{Z} \times \ldots \times \mathbb{Z}$ consisting of N-vectors $(n_1, \ldots, n_N)$ with integer coefficients. This space is not a vector space (because $\mathbb{Z}$ is not a field), in fact it is usually regarded as an <u>Abelian</u> <u>group</u> in which the "multiplication law" is just addition of integer vectors and the identity is the zero vector $(0, \ldots, 0)$.

[Note: Nakahara writes $\mathbb{Z}\oplus\mathbb{Z}$ instead of $\mathbb{Z}\times\mathbb{Z} = \mathbb{Z}^2$. It is just the space of integer vectors $(n_1, n_2)$, a lattice in the plane.].

---

Begin with another excursion into group theory. We'll only need Abelian groups for homology theory, but for now we'll consider some issues that apply to any group (Abelian or non-Abelian).

Recall that a group homomorphism is a map $f: G \to X$ between groups $G$ and $X$ such that $f(g_1)f(g_2) = f(g_1 g_2)$, $\forall\, g_1, g_2 \in G$.

Defs: $\ker f = \{ g \in G \mid f(g) = e_X \}$ $\qquad$ $e_X$ = identity element in $X$
$e_G$ = " " " $G$.

$\mathrm{im} f = \{ x \in X \mid x = f(g), \text{ some } g \in G \}$ (usual defn. of image).

Let $\quad K = \ker f \subset G$
$\quad\quad I = \mathrm{im}\, f \subset X.$ for brevity.

Thm: $\ker f = K$ is a normal subgroup of $G$.

Proof: ~~Let [illegible]~~. First show $K$ is a subgroup.
Let $k_1, k_2 \in K$. Then $f(k_1)f(k_2) = f(k_1 k_2)$

$$= e_X e_X = e_X,$$

so $k_1 k_2 \in K$.

Similarly show $K$ satisfies other axioms of a group.
Next show that $K$ is a normal subgroup. This means that $gkg^{-1} \in K$ for all $g \in G$, $k \in K$. Easy:

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)\, e_X\, f(g^{-1})$$
$$= f(g)f(g^{-1}) = f(gg^{-1}) = f(e_G) = e_X.$$

Hence $gkg^{-1} \in K$.

1/29/04

Note: Should have pointed this out earlier, but if $K$ is a normal subgroup of $G$, then the left cosets $gK$ and right cosets $Kg$ are identical (as subsets of $G$). This is what $gkg^{-1} \in K$ $\forall g \in G$, $k \in K$ means. If a subgroup is not normal, then the left and right cosets are generally different.

Thm: $\operatorname{im} f = I$ is isomorphic to the quotient group $G/K$,

$$\frac{G}{\ker f} \cong \operatorname{im} f. \quad (\text{The isomorphism is } [g] \mapsto f(g).)$$

Proof: First show that there is a 1-2-1 corresp. between cosets in $G/K$ and elements of $I$. To do this we need to show that 2 elements $g_1, g_2$ of $G$ map onto the same element of $I$ iff they belong to the same coset of $G/K$. So suppose $f(g_1) = f(g_2)$. This means $f(g_1) f(g_2)^{-1} = e_X = f(g_1 g_2^{-1}) \Rightarrow g_1 g_2^{-1} \in K \Rightarrow$ $g_1 = k g_2$ for some $k \Rightarrow g_1, g_2$ belong to same coset. Converse is easily proven. Thus $[g] \mapsto f(g)$ is a bijection.
Next we need to show that $[g] \mapsto f(g)$ is an isomorphism. This is easy. (That is, it's a homomorphism, since we already know it's a bijection.)

---

Now specialize to Abelian groups. Note, for an Abelian group, every subgroup is normal, so the quotient group is always defined.

For Abelian groups, convenient to change notation, use "+" for "group multiplication" etc. Table:

| general case | | Abelian |
|---|---|---|
| $xy$ | $\longrightarrow$ | $x+y$ |
| $x^{-1}$ | $\longrightarrow$ | $-x$ |
| $e$ | $\longrightarrow$ | $0$ |
| $x^n$ | $\longrightarrow$ | $nx$ |
| $\times$ | $\longrightarrow$ | $\oplus$ (Cartesian product) |

Examples of Abelian groups include:

$$\mathbb{R}^n, \; S^1, \; \mathbb{Z}, \; \mathbb{Z}_k$$

vector space $\quad \{e^{i\theta}\}. \quad$ integers modulo $k$ $= \{0, 1, \ldots, k-1\}$.

We will only be interested in discrete Abelian groups, which excludes things like $\mathbb{R}^n$ and $S^1$.

If $G$ is ~~an~~ a discrete Abelian group and $x_1, \ldots, x_r$ are elements of $G$ such that any $g \in G$ can be written in the form,

$$g = \sum_{i=1}^{r} n_i x_i, \qquad n_i \in \mathbb{Z},$$

then $G$ is said to be <u>generated</u> by the $\{x_i\}$ and the $\{x_i\}$ are said to be the <u>generators</u>. If $r < \infty$, then we say that $G$ is <u>finitely generated</u>. For homology theory we only need finitely generated Abelian groups. Notice that so far we're not saying that the generators are minimal in number (and in any case they are not unique).

Either the group elements $\sum_{i=1}^{r} n_i x_i$ for $n_i \in \mathbb{Z}$ are all unique, or there is some duplication (some group elements can be represented as a "linear combination" of the generators in more than one way).

In the former case we say that the group is <u>freely generated</u>, and that $G$ is a <u>free Abelian group of rank $r$</u>. In this case, every $g \in G$ can be <u>uniquely</u> represented as

$$g = \sum_{i=1}^{r} n_i x_i,$$

and $G$ is isomorphic to $\mathbb{Z}^r$, $G \cong \mathbb{Z}^r$, $g \mapsto (n_1, \ldots, n_r)$.

In effect, there is only one free Abelian group of rank $r$, it is $\mathbb{Z}^r$, and it can be visualized as the integer lattice in $r$-dimensional $\mathbb{R}^r$.

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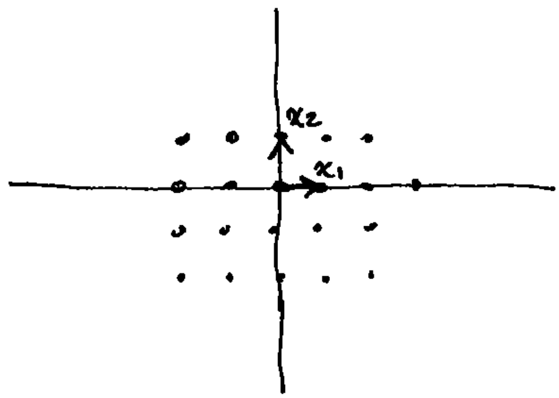
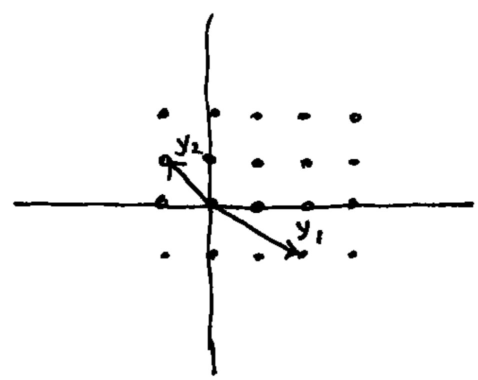
About the non-uniqueness of the generators. Take the case $r=2$ for illustration. Represent the generators $x_1, x_2$ as unit vectors in the plane, and the group as the lattice.




The "basis" $(x_1, x_2)$ spans the lattice, but it is not unique. We could use

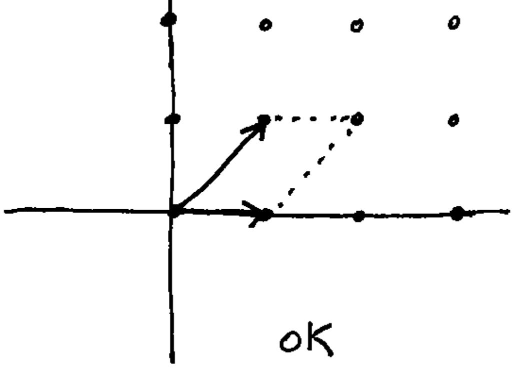
$$y_1 = 2x_1 - x_2$$
$$y_2 = -x_1 + x_2$$

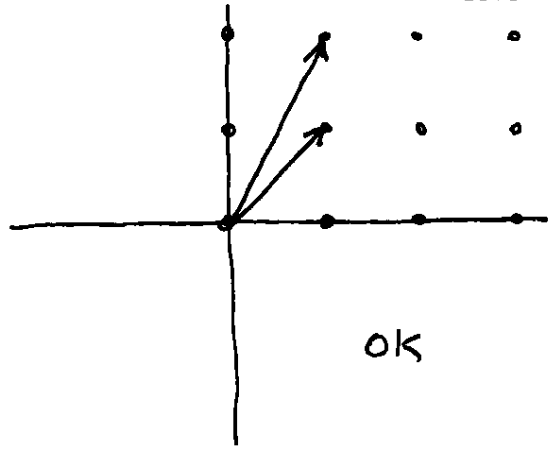


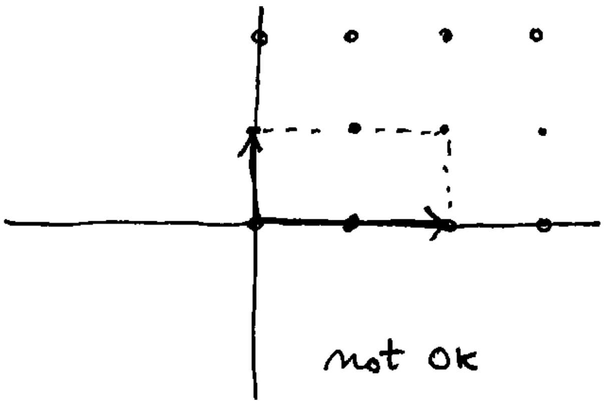

or $$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

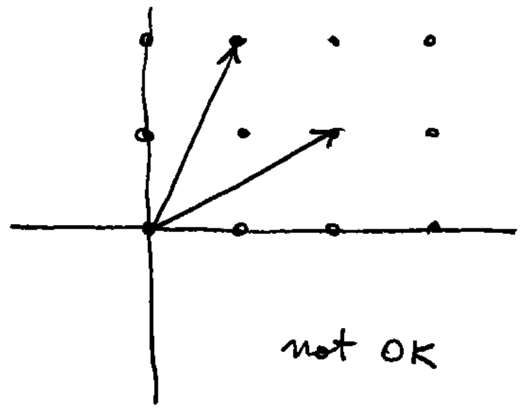
$\uparrow$ note $\det = +1$.

But not any linear combination will work. The new basis $(y_1, y_2)$ must span a cell that does contain any points inside or on the boundary, except at the 4 corners (= avg. of one point per cell, since each corner is shared among 4 cells).




The requirement for a valid change of basis is that the $r \times r$ matrix $M$,

$$\begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = M \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$$

must consist of integers and must have an inverse $M^{-1}$ that consists of integers. This means $M \in GL(r, \mathbb{Z})$. It also means that $\det M = \pm 1$.

---

Notice if we have a freely generated Abelian group of rank $r$ generated by $(x_1, \ldots, x_r)$, then the only way $\sum_{i=1}^{r} n_i x_i = 0$ is when $n_i = 0$, $\forall\, i$. Then we may borrow terminology from linear algebra and say that $(x_1, \ldots, x_r)$ are <u>linearly</u> <u>independent</u>.

1/29/04

To handle the general case (free or not free) of a finitely generated Abelian group, let $G =$ the group, $\{x_1, \dots, x_r\}$ a set of generators, and consider the map

$$f: \mathbb{Z}^r \to G : (n_1, \dots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i .$$

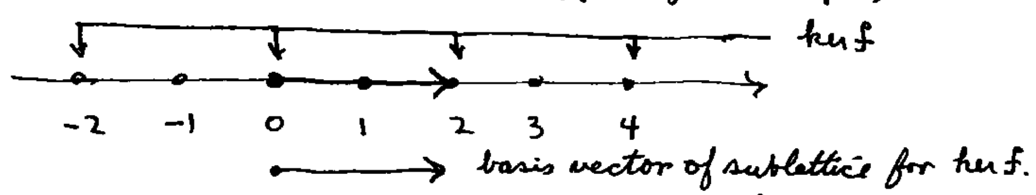
This map is <u>onto</u>, $\operatorname{im} f = G$, by the definition of "generators".
The condition that the group is free is precisely the condition $\ker f = \{(0, \dots, 0)\}$, that is, $\ker f$ is the trivial subgroup of $\mathbb{Z}^r$ containing the identity.
Then

$$G \cong \frac{\mathbb{Z}^r}{\ker f} \cong \mathbb{Z}^r, \quad \text{same conclusion as above.}$$

But if the group is not free, then $\ker f$ contains more than the identity element. In fact, it must be a <u>sublattice</u> of $\mathbb{Z}^r$, since it's closed under addition.

---

As an example, consider the case $r = 1$, so only one generator $x$. An Abelian group $G$ with one generator is called <u>cyclic</u>. If the group is free, then $G \cong \mathbb{Z}$. But suppose for example, that $2x = 0$.




Then $\ker f$ is the set $(\dots, -2, 0, 2, 4, \dots)$, the sublattice spanned by $(2)$. Call this subgroup $2\mathbb{Z}$.
↳ also sublattice.

[More generally, $k\mathbb{Z}$ is the set $\{kn \mid n \in \mathbb{Z}\}$ for $k \geq 1$]

Then $G = \dfrac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2 = \{0, 1\}$, integers modulo 2.

1/29/04

More generally, $\frac{\mathbb{Z}}{k\mathbb{Z}} \cong \mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ = integers modulo $k$.

We see that a cyclic group either contains an $\infty$ number of elements, in which case it is isomorphic to $\mathbb{Z}$, or else it contains a finite number $k \geq 1$ elements, in which case it is isomorphic to $\mathbb{Z}_k$.

---

Now we quote the facts (without proof) for the case of arbitrary $r$.

As above, $G$ = finitely generated Abelian group with generators $\{x_1, \ldots, x_r\}$, and $f: \mathbb{Z}^r \to G : (n_1, \ldots, n_r) \mapsto \sum_{i=1}^{r} n_i x_i$. Again, $\operatorname{im} f = G$.

Fact 1. Any subgroup of $\mathbb{Z}^r$ is a sublattice of $\mathbb{Z}^r$ which can be spanned by some set of integer vectors (lin. independent) (elements of $\mathbb{Z}^r$), call them $\{y_1, \ldots, y_p\}$, $p \leq r$. In particular, $\ker f$ can be written,

$$\ker f = \left\{ \sum_{i=1}^{p} m_i y_i \;\middle|\; m_i \in \mathbb{Z} \right\}.$$

This is the general form of a finitely generated Abelian group.

Fact 2.
$$\frac{\mathbb{Z}^r}{\ker f} \cong \underbrace{\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_p}}_{p \text{ factors}} \times \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{r-p \text{ factors}} \cong G.$$

$$k_i \geq 1, \quad i = 1, \ldots, p.$$

Note that $\mathbb{Z}_1$ (the case $k=1$) is just the trivial group $\{0\}$ (the cyclic group with one element); if this occurs in the list it can be dropped. Note also that we never had to say that the generators $\{x_1, \ldots, x_r\}$ were "minimal"; if a smaller set of generators would work, then there will appear $\mathbb{Z}_1$ factors in the final product.