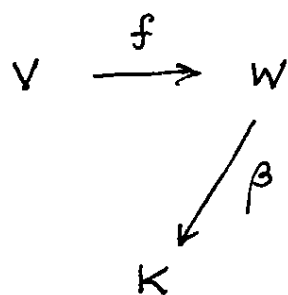


Begin with relation between dual spaces  $V^*$  and  $W^*$  when we have a map  $f: V \rightarrow W$ . (There is still no metric.) Does  $f$ , which takes a vector  $v \in V$  and produces a vector  $w = f(v) \in W$ , do something similar to  $V^*$  and  $W^*$ , that is, take a form (= dual vector) in  $V^*$  and produce another form in  $W^*$ ? The answer is no, in general. But it does allow one to take a form in  $W^*$  and produce another form in  $V^*$  (in the reverse direction from the action of  $f$  itself). This action on forms, going from  $W^* \rightarrow V^*$ , is called the pull-back.

Suppose we are given  $f: V \rightarrow W$  and some  $\beta \in W^*$ , that is  $\beta: W \rightarrow K$  (the scalars). Picture of the maps:



The picture makes it obvious that we can go directly from  $V$  to  $K$  by composing the maps, that is, let  $\alpha \in V^*$  be defined by

$$\alpha = \beta \circ f, \quad \alpha: V \rightarrow K.$$

This specifies a mapping between  $W^*$  and  $V^*$  which we denote by  $f^*: W^* \rightarrow V^*$ , called the pull-back of  $f$ . That is,

$$\boxed{f^*: W^* \rightarrow V^*: \beta \mapsto \beta \circ f.}$$

An equivalent definition of the pull back is to specify  $f^*\beta$  by its action on vectors in  $V$ :

$$\boxed{(f^*\beta)(v) = \beta(f(v)), \quad \forall v \in V} \quad \text{defines } f^*\beta.$$

There is no natural way to define a map:  $V^* \rightarrow W^*$  (a "push-forward") unless  $f$  is invertible, whereupon you could use  $f^{-1*}$ .

We've done almost everything that can be done without a metric, so now let's introduce one. Nakahara's discussion of this is backwards, confused, and wrong in part, so ignore what he says and use the following.

Begin with the case  $K = \mathbb{R}$  (real vector spaces), since things are somewhat more complicated when  $K = \mathbb{C}$ . Idea of metric is measure of distance.

Given real vector space  $V$ . A metric or (metric tensor) is a map

$$g: V \times V \rightarrow \mathbb{R},$$

such that:

1)  $g$  is linear in both operands,

$$\left. \begin{aligned} g(c_1 u_1 + c_2 u_2, v) &= c_1 g(u_1, v) + c_2 g(u_2, v) \\ g(u, c_1 v_1 + c_2 v_2) &= c_1 g(u, v_1) + c_2 g(u, v_2) \end{aligned} \right\} \begin{aligned} \forall u_1, u_2, u \in V \\ \forall v_1, v_2, v \in V \\ \forall c_1, c_2 \in \mathbb{R}. \end{aligned}$$

2)  $g$  is positive definite,

$$g(v, v) \geq 0, \quad \forall v \in V$$

$$g(v, v) = 0 \quad \text{iff} \quad v = 0$$

3)  $g$  is symmetric,

$$g(u, v) = g(v, u), \quad \forall u, v \in V.$$

Then the quantity  $g(u, v)$  is the inner product of 2 vectors, which we may denote by  $\langle u, v \rangle$  (another notation for it).

Let  $\{e_i\}$  be a basis in  $V$ . Then we define

$$g_{ij} = g(e_i, e_j) = \text{component matrix of } g \text{ in the given basis.}$$

Condition 2 implies that  $g_{ij}$  is a positive definite matrix, hence that  $\det g_{ij} \neq 0$  ( $g_{ij}$  is nonsingular, since all of its eigenvalues are positive.)

Note that in relativity theory, we deal with metrics that are not positive definite. In this case, we replace requirement 2) in the definition with the nonsingularity requirement, that  $\det g_{ij} \neq 0$ . (This condition makes reference to a basis, but is independent of the basis chosen.) A way of writing the nonsingularity condition without reference to a basis is to say, ~~if~~ if  $g(u, v) = 0$  for all  $u \in V$ , then  $v = 0$ .

A metric, regarded as a distance function on  $V$ , induces an association between  $V$  and  $V^*$ . The latter is an alternative way of looking at a metric. In the expression  $g(u, v)$ , regard  $u$  as fixed and  $v$  as variable. To emphasize this, write

$$g_u(v) = g(u, v),$$

thereby defining a function  $g_u: V \rightarrow \mathbb{R}$ . Such a function is a form, i.e.,  $g_u \in V^*$ . Thus we have a  $\neq$  mapping,  $\bullet$

$$g: V \rightarrow V^*: u \mapsto g_u. \quad (\text{linear})$$

It's abuse of notation to use the same symbol  $g$  for this map as for the distance function, but they're so closely related that everyone does so anyway. Now put this into component language.

Let  $v \in V$ , let  $\{e_i\}$  be a basis in  $V$  so that  $v = \sum_{i=1}^n v^i e_i$ , and let  $\alpha = g_v$ .

Let  $u \in V$ , let  $\{e_i\}$  be a basis in  $V$  and write  $u = \sum_{i=1}^n u^i e_i$ , and let  $\alpha = g_u \in V^*$ . Find components  $\alpha_i$  of  $\alpha$  w.r.t. the dual basis.

$$\alpha(v) = \sum_j \alpha_j v^j = g_u(v) = \sum_{ij} u^i g_{ij} v^j, \quad \forall v \in V$$

so 
$$\alpha_j = \sum_i u^i g_{ij}.$$

The form  $\alpha = g_u$  is so closely associated with  $u$  that in applications it is often identified with it, and we just write  $u_i$  (with a lower index) instead of  $\alpha_i$  or  $(g_u)_i$ . This is called lowering an index.

$$u_i = \sum_j g_{ij} u^j.$$

Now because  $g_{ij}$  is invertible, we ~~can~~ have the inverse map:  $V^* \rightarrow V$ .

Let  $g^{ij}$  be the inverse matrix of  $g_{ij}$  (std notation), so that

$$\sum_k g_{ik} g^{kj} = \delta_i^j.$$

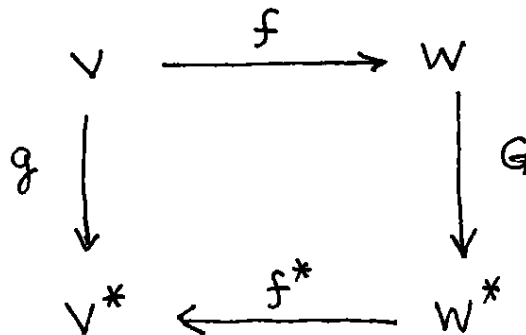
Then the inverse map  $g^{-1}: V^* \rightarrow V: \alpha \mapsto u$  is specified in components by

$$u^i = \sum_j g^{ij} \alpha_j.$$

One often writes  $\alpha^i$  instead of  $u^i$  (same symbol, but with an upper index) and speaks of raising an index.

Summary: A metric  $g$  on  $V$  induces an isomorphism between  $V$  and  $V^*$ .  
↙ real vector space. (linear)

Now examine how metrics interact with maps. Suppose we have a linear map between spaces, each of which possesses a metric. Let  $g$  be the metric on  $V$  and  $G$  the metric on  $W$ , and suppose  $f: V \rightarrow W$  is linear. Then we have the following picture of maps and spaces,



Notice that there is a route to get from  $W$  to  $V$ , since  $g$  and  $G$  are invertible.

Leads to definition,  $\tilde{f}: W \rightarrow V$ , the adjoint of  $f$ , by

$$\tilde{f} = g^{-1} f^* G. (= g^{-1} \circ f^* \circ G).$$

This is equivalent to

$$g(\tilde{f}w, v) = G(w, fv), \quad \forall v \in V, w \in W,$$

or

$$\langle \tilde{f}w, v \rangle_g = \langle w, fv \rangle_G, \quad \text{which should look familiar.}$$

(exercise to show this).

Now what changes when you go to metrics on complex vector spaces.

Now  $g: V \times V \rightarrow \mathbb{C}$ , such that:

1)  $g$  is linear in 2nd operand, and anti-linear in the first,

$$g(c_1 u_1 + c_2 u_2, v) = \bar{c}_1 g(u_1, v) + \bar{c}_2 g(u_2, v)$$

$$g(u, c_1 v_1 + c_2 v_2) = c_1 g(u, v_1) + c_2 g(u, v_2)$$

2)  $g(v, v) = \text{real}, > 0, \forall v \in V$

$$g(v, v) = 0 \text{ iff } v = 0$$

3)  $g(u, v) = \overline{g(v, u)}$ .

(overbar = complex conjugate).

The associated mapping  $g: V \rightarrow V^*$  is defined as before,

$$u \mapsto g_u, \quad g_u(v) = g(u, v) = \langle u, v \rangle,$$

but it is now an antilinear map (point missed by Nakahara).

The adjoint is defined as above,  $\tilde{f} = g^{-1} f^* G$ . Note that it is linear, since  $g^{-1}$  and  $G$  are antilinear. (Usual notation in QM,  $\tilde{f} = f^\dagger$ ).

Note: usually in QM when we talk about the adjoint of a linear operator, we are thinking of the case  $W=V$ , so  $G=g$ , and so  $\tilde{f} = g^{-1}f^*g$ , and

$$\langle \tilde{f}u, v \rangle = \langle u, fv \rangle.$$

Next, tensors. ( $K = \mathbb{R}$  here).

A tensor  $T$  of type  $(p, q)$  is a multilinear map,

$$T: \underbrace{V^* \times \dots \times V^*}_{p \text{ times}} \times \underbrace{V \times \dots \times V}_{q \text{ times}} \rightarrow \mathbb{R}$$

Examples:

A covector  $\alpha \in V^*$  is a tensor of type  $(0, 1)$ , since  $\alpha: V \rightarrow \mathbb{R}$ .

A vector  $v \in V$  is considered a tensor of type  $(1, 0)$ , since it can be considered a map  $v: V^* \rightarrow \mathbb{R}: \alpha \mapsto \alpha(v) \equiv v(\alpha)$ .

$g: V \times V \rightarrow \mathbb{R}$  is a tensor of type  $(0, 2)$

$g^{-1}: V^* \times V^* \rightarrow \mathbb{R}$  " " " " "  $(2, 0)$

etc.

There are 2 operations on tensors, the tensor product and the contraction.

let  $\mu =$  tensor of type  $(p, q)$

$\nu =$  " " "  $(r, s)$

then  $\mu \otimes \nu =$  " " "  $(p+r, q+s)$ .

Def:  $\mu \otimes \nu (\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_r; v_1, \dots, v_q, u_1, \dots, u_s)$

$$= \mu(\alpha_1, \dots, \alpha_p; v_1, \dots, v_q) \nu(\beta_1, \dots, \beta_r; u_1, \dots, u_s).$$

The contraction takes a tensor of type  $(p, q)$  and produces one of type  $(p-1, q-1)$ .

Illustrate by contracting on 1st slots. Let  $\mu =$  tensor of type  $(p, q)$ .

$$(\text{Contracted } \mu)(\alpha_2, \dots, \alpha_p; \nu_2, \dots, \nu_q) = \sum_i \mu(e^{i*}, \alpha_2, \dots, \alpha_p; e_i, \nu_2, \dots, \nu_q)$$

where  $\{e_i\}$  is a basis in  $V$  and  $\{e^{i*}\}$  is dual basis in  $V^*$ .

The contraction is independent of the basis (but does in general depend on which slots are contracted.)

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Now some background on spaces, manifolds, topology, etc. First some overview of the hierarchy of spaces you get as you add more and more structure.

At the most primitive level, a "space" is just a set of objects we call "points" without any additional structure.

To this we may add the structure required to give the space a topology. Intuitively, topology tells you something about which points are "close" to one another, so that you can think about "neighborhoods" of points and how different neighborhoods fit together to make the whole space.

It would seem that the concept of "closeness" requires a definition of "distance" between points, and, indeed, if we impose a distance function or metric on our space, it becomes possible to talk about topology. Thus, metric spaces are always topological spaces. (But don't confuse a metric in the sense of a distance function with a metric tensor — the latter requires much more structure before it can be defined.)

It turns out however, that topological matters can always be expressed in terms of only the open sets of the space (that is, open subsets), without reference to "distances".

For example, if we have a map  $f: X \rightarrow Y$  between topological spaces  $X, Y$  (defined momentarily), then it turns out that  $f$  is continuous if the inverse image of open sets (in  $Y$ ) are open sets (in  $X$ ). This

defn of continuity makes no reference to distance, only the open sets. nevertheless, if  $X$  and  $Y$  do possess a distance function, then this defn of continuity coincides with the familiar one based on distances ( $\epsilon$  and  $\delta$  defn.) (The latter intuitively says that if two values of the independent variable are close together, then the two values of the dependent variable are close together.) Official defn:

Def. Let  $X$  be a space (set) and  $\{U_i, i \in I\}$  be a collection of subsets. ~~Let~~ Let the set  $\{U_i\}$ : 1) contain  $\emptyset$  and  $X$ ; 2) be closed under all unions and finite intersections. Then the subsets  $U_i$  are said to be open sets, and  $X$  is said to be a topological space.

on  $\mathbb{R}$ , the usual topology is obtained by defining the open sets to be open intervals and their unions. Similarly for  $\mathbb{R}^n$ . Also leads to a usual topology on subsets of  $\mathbb{R}^n$ .

Thus, given a set  $X$  you can in general define more than one distinct topology on it. But in this course we'll mostly work with  $\mathbb{R}^n$  or subsets of  $\mathbb{R}^n$ , where the topology is the usual one.

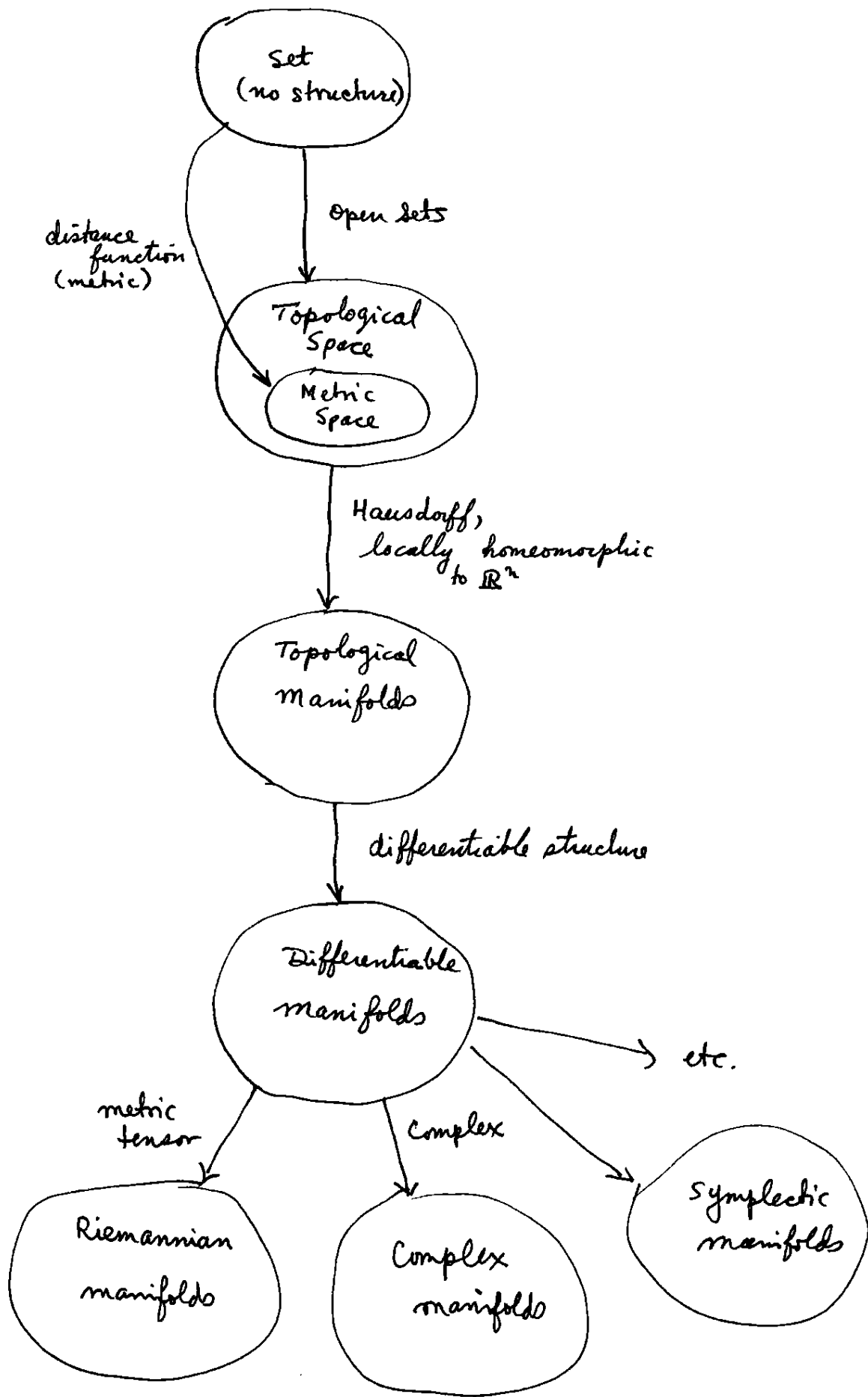
The usual topology on  $\mathbb{R}^n$  has the Hausdorff property (see book for defn), so in this course we will assume that the Hausdorff property holds (usual in physical applications).

Next, a topological space that is both Hausdorff and locally homeomorphic to  $\mathbb{R}^n$  (terminology explained momentarily), is a topological manifold.

At this level we can talk about continuity but not differentiability. The latter requires additional structure, to turn a topological manifold into a differentiable manifold.

Finally, given a differentiable manifold, we can add additional structure ~~to get~~ (such as a metric tensor) to get a Riemannian manifold, or a symplectic manifold, complex manifold, etc.





Let's look at some of the structure that exists at the level of a topological space.

Let  $X =$  a topological space.

Def. A subset  $A \subset X$  is closed if the complement  $X - A$  is open.

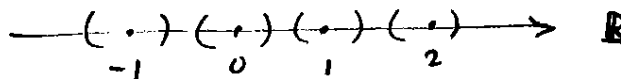
Also define closure, interior (see text).

Definition of continuity given above.

Now compactness, a property that is a little mysterious if you've never taken a course in topology.

Def. A subset  $A \subset X$  of a topological space is compact if every open cover contains a finite subcover. (An open cover is a set of open sets whose union contains  $A$ .)

Example,  $\mathbb{Z} \in \mathbb{R}$ .



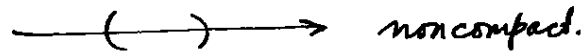
$X = \mathbb{R}$   
 $A = \mathbb{Z}$

Open cover has  $\infty$  # of open intervals, surrounding each integer. Remove any one and  $\mathbb{Z}$  is no longer covered. Therefore  $\mathbb{Z}$  is not compact in  $\mathbb{R}$ .

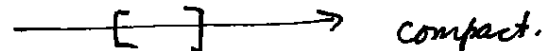
Important thm. for compactness in  $\mathbb{R}^n$  (Heine-Borel):

Thm. A subset of  $\mathbb{R}^n$  is compact if and only if it is closed and bounded.

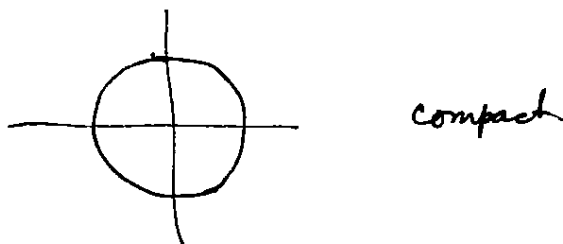
Examples: 1) Open interval in  $\mathbb{R}$



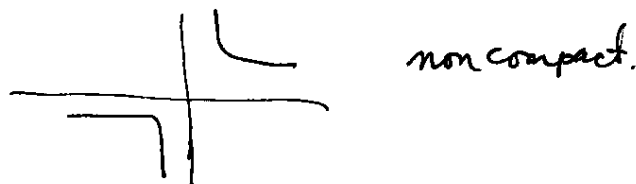
2) Closed " " "



3)  $S^1$  (circle) in  $\mathbb{R}^2$ :



4) Hyperbola in  $\mathbb{R}^2$



Mention compactness has on group representation theory, e.g.

compact groups such as  $SU(n)$ ,  $SO(n)$ , etc.

v.s. noncompact ones like Lorentz,  $GL(n, \mathbb{R})$ ,  $Sp(2n)$  etc.

Now, Connectedness. There are 2 kinds of notions, connected and arc-wise connected, which are the same in most physical applications. Idea corresponds to intuitive notion of connectedness. official defns:

A topological space  $X$  is connected if it cannot be written as the disjoint union of two open sets.

It is arcwise connected if for  $\forall x, y \in X$ , there exists a continuous curve in  $X$  connecting  $x$  and  $y$  (i.e., a continuous map:  $[0, 1] \rightarrow X$  with  $f(0) = x$  and  $f(1) = y$ .)

Finally,  $X$  is simply connected if every closed loop can be continuously contracted to a point.

Now we turn to the principal notion of topological equivalence, namely, the homeomorphism.

Def. Given two topological spaces  $X$  and  $Y$ . A map  $f: X \rightarrow Y$  is said to be a homeomorphism if it is continuous and possesses a continuous inverse  $f^{-1}: Y \rightarrow X$ . ~~has an~~ If such an  $f$  exists,  $X$  and  $Y$  are said to be homeomorphic.

Nakahara discusses this concept in the framework of a continuous deformation of one space into another (the coffee mug to the doughnut, for example). Continuous deformation, however, requires an embedding space (in which  $X$  and  $Y$  are subsets), and this plays no part in the definition of homeomorphism.

topological

A central problem of topology is to classify spaces, to within a homeomorphism. One can easily show that the relation "homeomorphic" is an equivalence relation, thus topological spaces are divided into equivalence classes.

It is not known how to classify all equivalence classes of topological spaces. That is, given two spaces, it may not be easy to show that they are homeomorphic, apart from finding the homeomorphism that connects them. However, it may be relatively easy to show that they are not homeomorphic, by using topological invariants.

A topological invariant is a quantity or characteristic that is invariant under homeomorphisms. Thus, if two spaces have different invariants, they are not homeomorphic.

see book for more on topological invariants, Euler characteristic