(0)

1/22/04

Correction for last lecture: A linear homomorphism satisfies

$$f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2)$$

where $f: V \to W$, V, W vector spaces, c_1, c_2 scalars, $v_1, v_2 \in V$.

This is not a special case of homomorphisms between groups.

Now continue with examples of equivalence relations and quotient classes...

---

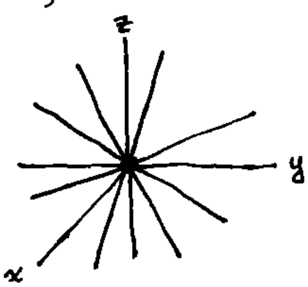
which comes up in Nakahara's examples.

Another note: definition of $\mathbb{RP}^n$ (n-dim'l real projective space).

Define an equivalence relation in $\mathbb{R}^{n-1} \setminus \{0\}$ by requiring

$x \sim y$ if $x = cy$, c = real, $c \neq 0$, $x, y \in \mathbb{R}^{n+1}$, $x \neq 0, y \neq 0$,

that is, if non-zero vectors x and y are proportional (note c may be negative). The equivalence classes are two-sided (nondirected) lines through the origin, with the origin itself removed. Picture in $\mathbb{R}^3$ ($n=2$):



$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\sim \text{(proportionality)}}$$

$\mathbb{RP}^n$ is the family of the 2-sided lines.

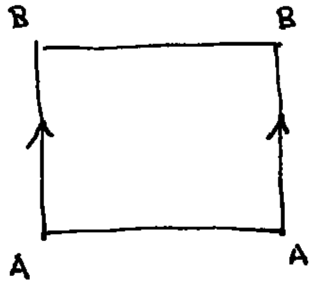
Each 2-sided line intersects the sphere S^n in 2 points (antipodal points), so another way to write $\mathbb{RP}^n$ is

$$\mathbb{RP}^n = \frac{S^n}{\sim \text{(antipodal)}}.$$

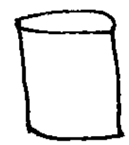
$\mathbb{RP}^2$ is equivalent to a disk (D^2) with opposite points on the boundary identified, which can be identified with the northern hemisphere + equator of S^2 (with opposite points on the equator identified.)

Most of the examples so far involve equivalence relations that are produced by group actions. (More on group actions later.) Here are some equivalence relations that are not produced by group actions.

(7) X = square with edges identified. (Equivalence classes imply glueing rules.)

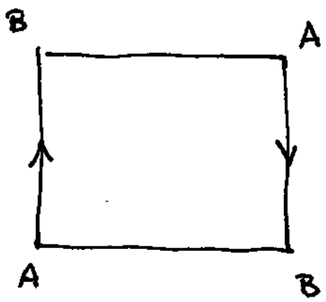


$= \frac{X}{\sim} =$

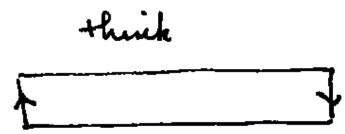


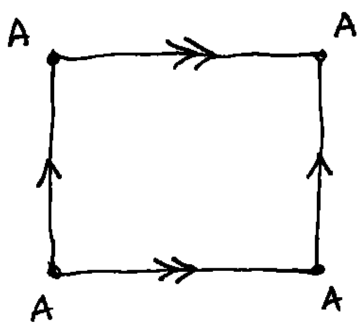
segment of cylinder.

(Each point $x \in X$ is in an equivalence class by itself, except for points on the right and left sides, which are identified in pairs.) Variations on this:



= Möbius strip



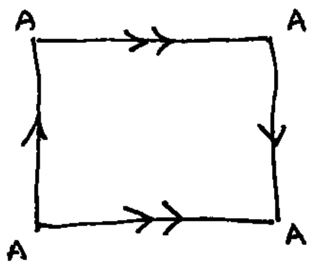


=

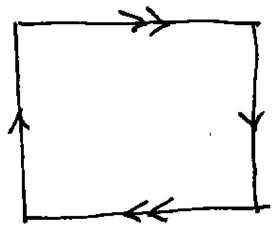
= 2-torus, T^2

← points on corners are in a 4-point equiv. class. Other points on sides are in 2-point equiv. classes. Other (interior) points are in 1-point equiv. classes.

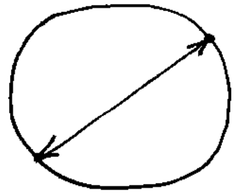
1/22/04



= Klein bottle. a 2-dim. surface that cannot be imbedded in $\mathbb{R}^3$ without ~~sf~~ self-intersections.



=

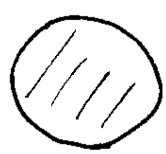


= $\mathbb{R}P^2$

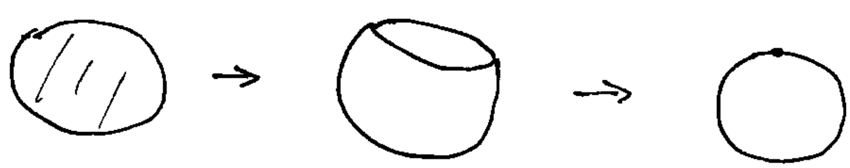
disk with opposite points on boundary identified

<u>Note</u>: the 2-disk D^2 is the set $x^2+y^2 \leq 1$ in the (x-y) plane, ie it is the interior of a circle <u>plus</u> the boundary.

(B) A different identification for the disk D^2. Let interior points be in equivalence classes, let all boundary points be in a single equiv. class.

$X = D^2 =$  . Let $x \sim y$ if $|x| = |y| = 1$.

Then

$\frac{X}{\sim} = \frac{D^2}{\sim}$,  $= S^2$.

$$\frac{D^2}{\sim} = S^2.$$

1/22/04

(9) Example from abstract group theory. First, defn of group.

G = a set is a <u>group</u> if there is a "multiplication" rule defined on G such that:

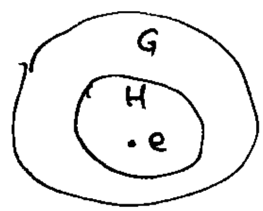
a) If $g, h \in G$, then $gh \in G$

b) $\exists\ e \in G$ (the identity) such that $eg = ge = g$, $\forall\, g \in G$.

c) If $g \in G$ then $\exists\ g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$

d) If $g, h, k \in G$, then $(gh)k = g(hk)$.

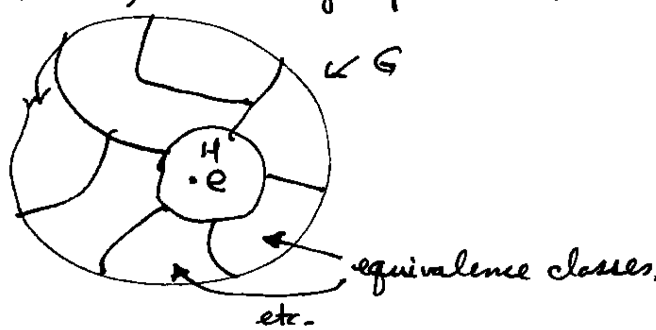
<u>Abelian</u> means <u>commutative</u> when speaking of groups.
Now let G = group, $H \subset G$ a subgroup.



Let $g, g' \in G$. Let $g \sim g'$ if $\exists\ h \in H$ such that $g = g'h$.
Easily show this is an equivalence relation. The equivalence class,

$$[g] = \{ gh \mid h \in H \}$$ is denoted gH and called a <u>left</u> <u>coset</u>.

The equivalence relation divides G into disjoint equivalence classes (the left cosets); can easily show, the subgroup $H = [e]$ is one of them. Picture:



Quotient space is $$\frac{G}{H} = \text{space of left cosets.}$$

<u>Basic fact:</u>
$\frac{G}{H}$ not a group unless H = <u>normal subgroup</u>, means $gHg^{-1} = H$, $\forall\, g \in G$.

Then $\frac{G}{H}$ is called the <u>quotient group</u>

1/22/04

Elaborate on this. We want to define multiplication of left cosets. Denote the multiplication law by $*$. How to define $[a]*[b]$, for $a,b \in G$? Logical choice is

$$[a]*[b] = [ab].$$

But is this meaningful? Problem is that a, b are only representative elements of cosets, we must make sure answer is independent of which rep. element we choose. Let $a' = ah_1$, $b' = bh_2$ be other choices of rep. elements of $[a]$, $[b]$, where $h_1, h_2 \in H$. We demand that $a'b' = abh_3$, where $h_3 \in H$, in order for mult. law $*$ to be meaningful. In other words, mult.-law is meaningful if for all $a, b \in G$ and $h_1, h_2 \in H$, $\exists\ h_3 \in H$ such that

$$ah_1 bh_2 = abh_3$$

$$\Leftrightarrow \quad h_1 b h_2 = bh_3$$

$$\Leftrightarrow \quad h_1 b = b h_4 \qquad \text{where} \quad h_4 = h_3 h_2^{-1}$$

$$\Leftrightarrow \quad bh_4 b^{-1} = h_1$$

$$\Leftrightarrow \quad \text{for all } b \in G,\ h \in H, \qquad bhb^{-1} \in H.$$

← A simpler stmt of the condition that the mult. law be meaningful

The latter condition is also written $bHb^{-1} = H$, (for $\forall b \in G$.) If true, H is called a <u>normal</u> <u>subgroup</u>, and $*$ of cosets is defined.

Remains to be shown that $\frac{G}{H}$ with $*$ as mult. law satisfies group axioms (exercise for you).

<u>Note</u>, if G is <u>Abelian</u>, then $gHg^{-1} = Hgg^{-1} = H$, so all subgroups are normal and quotient group is always defined.

Now for some selected topics in <u>linear algebra</u>, with a slight geometrical flavor.

Let V be a vector space over a field K. In practice, K usually = $\mathbb{R}$ or $\mathbb{C}$ (real or complex vector spaces); this means that the coefficients used in forming linear combinations of vectors (the <u>scalars</u>) are either real or complex numbers. (In general it is not meaningful to say whether the vectors themselves are "real" or "complex".)

Assume you know definitions of <u>vector space</u>, <u>basis</u>, <u>linear (in)dependence</u>, <u>span</u>, <u>dimension</u>, etc. We will (for now) deal only with finite dimensional vector spaces.

<u>Psychological problem</u>: Physicists have a tendency to assume that all vector spaces possess a <u>metric</u>, i.e., a definition of a scalar product, but for many problems the vector spaces you encounter do not possess any metric that is natural to the problem at hand. (Example: the (x,p) phase space of a mechanical problem in 1D.) Of course you can always introduce an arbitrary metric, but this is almost always a bad idea unless the metric is a natural outcome of the structure presented by your problem, or (occasionally) unless you can show that results don't depend on the choice of the metric.

Instead it is better to develop those structures of linear algebra that can be developed without reference to any metric, and to understand those. <u>Then</u> we introduce a metric and see what new structures become available. This is what we shall do.

⑥ 1/22/04

First, let V be a vector space and $\{e_1, e_2, \ldots, e_n\}$ be a basis in V (thus, $\dim V = n$). If $v \in V$, then we can write in a unique way,

$$v = \sum_{i=1}^{n} v^i e_i ,$$

where $v^i \in K$ are the components of v w.r.t. the basis $\{e_i\}$. The upper (contravariant) index on v is deliberate. Note, $v^i \in K$ but $e_i \in V$.

Now consider linear maps, that is, linear homomorphisms between vector spaces, $f: V \to W$. These satisfy

$$f(v_1 + v_2) = f(v_1) + f(v_2), \quad \forall\, v_1, v_2 \in V.$$
$$f(kv) = k f(v) \quad \forall\, v \in V,\ k \in K \text{ (the field)}$$

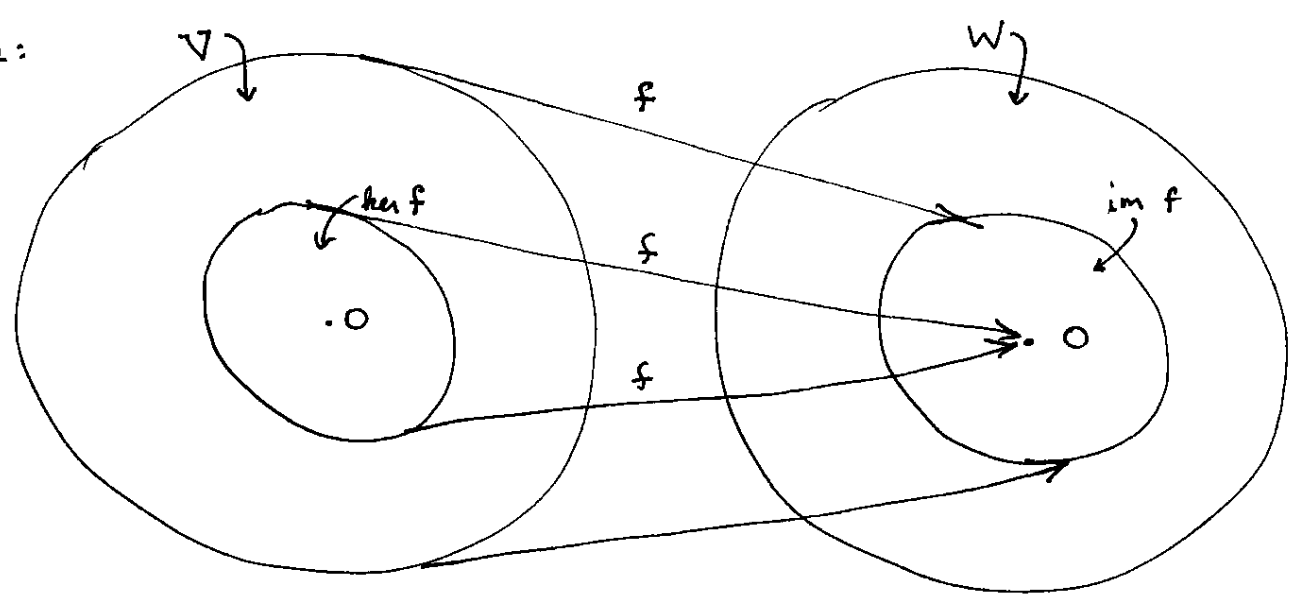
There are two spaces that can be defined using only the linear structure:

$$\ker f = \{ v \in V \mid f(v) = 0 \} = \text{set of all vectors in } V \text{ annihilated by } f$$

$$\operatorname{im} f = \{ w \in W \mid w = f(v) \text{ for some } v \in V \} = \text{usual defn. of image.}$$

Note, $\ker f \subset V$, $\operatorname{im} f \subset W$, note also, $\ker f$ is a vector subspace of V, $\operatorname{im} f$ is a vector subspace of W

Picture:



Basic theorem (important):

$$\boxed{\dim \ker f + \dim \operatorname{im} f = \dim V}$$

Context: $f: V \to W$, linear map. field K ($=\mathbb{R}$ or $\mathbb{C}$)

<u>Proof:</u> Let $\{g_1, \ldots, g_r\}$ be a basis for $\ker f$ ($g_i \in V$)
Let $\{h_1', \ldots, h_s'\}$ be a basis for $\operatorname{im} f$ ($h_i' \in W$).

Let v = any vector in V. Note $f(v) \in \operatorname{im} f$, so it can be expanded in the basis $\{h_i'\}$:

$$f(v) = \sum_{i=1}^{s} c^i h_i', \quad c^i = \text{expansion coefficients}, \quad c^i \in K.$$

But for each i, $\exists$ some vector $h_i \in V$ such that $h_i' = f(h_i)$ ($f: h_i \mapsto h_i'$) (h_i is not unique, in general). So

$$f(v) = \sum_{i=1}^{s} c^i f(h_i) = f\left(\sum_{i=1}^{s} c^i h_i\right),$$

or $f\left(v - \sum_{i=1}^{s} c^i h_i\right) = 0$. Thus $v - \sum_{i=1}^{s} c^i h_i \in \ker f$,

and this vector can be expanded in the basis $\{g_i\}$, say $\sum_{i=1}^{r} d^i g_i$, $d^i \in K$

Thus, $$v = \sum_{i=1}^{r} d^i g_i + \sum_{i=1}^{s} c^i h_i,$$

and we see that any $v \in V$ can be written as a lin. comb. of $\{g_i\}, \{h_i\}$.

Now, are these vectors lin. indep? To show that they are, let

$$\sum_{i=1}^{r} a^i g_i + \sum_{i=1}^{s} b^i h_i = 0.$$

Apply f to both sides, use fact that $f(g_i) = 0$, $f(h_i) = h_i'$. Gives

$$\sum_{i=1}^{s} b^i h_i' = 0 \Rightarrow b^i = 0 \text{ since } \{h_i'\} \text{ are lin. indep.}$$

But this $\Rightarrow \sum_{i=1}^{r} a_i g_i = 0 \Rightarrow a^i = 0$ since the $\{g_i\}$ are lin indep. So all coeffs a^i, b^i vanish, and the set $\{g_i, h_i\}$ are lin. indep. and span V (they form a basis for V).

Thus, $\dim V = r+s = \dim \ker f + \dim \operatorname{im} f$. QED

Some intuition about this theorem: f acts on V and annihilates some vectors (those in $\ker f$). The ones it doesn't annihilate go into $\operatorname{im} f$. Therefore $\dim V = \dim \ker f + \dim \operatorname{im} f$. (Or a way of remembering this thm.)

Remark: Nakahara calls the subspace of V spanned by $\{h_i\}$ the "orthogonal complement" to $\ker f$, and he writes it $(\ker f)^\perp$. Please ignore this. We don't have a metric on our vector spaces yet, so "orthogonal" is undefined. In any case, the vectors $\{h_i\}$ are not unique, because you can add any element of the kernel to them without changing their definition. (That is, if $k \in \ker f$, then $f(h_i + k) = f(h_i) = h_i'$, so $h_i + k$ works just as well as h_i). So the "space spanned by $\{h_i\}$" has no invariant meaning.

Another remark: It turns out, however, that there is a way of defining a space that is "complementary" to $\ker f$, in a certain sense, but it is not a subspace of V, it is a quotient space. See the HW problems.

~~Now the concept of the dual space to a vector space V. Here we assume V is finite dimensional, and we let field $K = \mathbb{C}$ to be concrete (this subsumes the case $K = \mathbb{R}$).~~

~~Consider the special case of a linear map,~~

~~$$\alpha: V \to \mathbb{C},$$~~

~~i.e. where $W = \mathbb{C}$. Such a map is called a dual vector or covector on V~~

Now we move to the concept of the dual space to a vector space V. Setup: Let K be the field over which V is defined ($K = \mathbb{R}$ or $\mathbb{C}$ in practice). Elements of K are called "scalars". We assume V is finite-dimensional.

Let α be a linear map, $\alpha: V \to K$. This is a special case of a linear map, where $W = K$ (a one-dimensional space).

Such maps are called dual vectors or covectors.

Def. The dual space to V, denoted V^*, is the set of all such linear maps (dual vectors).

$$V^* = \{ \alpha \mid \alpha: V \to K, \text{ linear} \}.$$

First note that V^* (like V) is a vector space, under the obvious definition of multiplication of maps by scalars and the addition of maps:

For $\alpha_1, \alpha_2 \in V^*$ and $c_1, c_2 \in K$, define $c_1\alpha_1 + c_2\alpha_2$ by

$$(c_1\alpha_1 + c_2\alpha_2)(v) = c_1\, \alpha_1(v) + c_2\alpha_2(v), \qquad v \in V.$$

Now it turns out that $\boxed{\dim V = \dim V^*}$ (important fact). Easy way to see this:

Let $\dim V = n$, let $\{e_i,\ i = 1, \dots, n\}$ be a basis in V.

Define $\alpha_i = \alpha(e_i)$, call $\{\alpha_i\}$ the components of $\alpha: V \to K$.

For given α, the components α_i uniquely specify α. That is, if the n scalars $\alpha_i \in K$ are given, then the action of α on any vector $v \in V$ is determined:

$$\alpha(v) = \alpha\left(\sum_{i=1}^{n} v^i e_i \right) = \sum_{i=1}^{n} v^i\, \alpha(e_i) = \sum_{i=1}^{n} v^i \alpha_i$$

Conversely, if α is given, the α_i are determined by $\alpha_i = \alpha(e_i)$. So this provides a 1-to-1 map between V^* and K^n ($= \mathbb{R}^n$ or $\mathbb{C}^n$). Moreover, this map is linear (hence a linear isomorphism). Hence $\dim V^* = n = \dim V$.

Remark 1: The lower index on α_i is deliberate, just as is the upper index on v^i: the lower index is a "covariant" index and the upper is "contravariant". (Standard terminology in tensor analysis.)

Remark 2: The concept of the dual space is important. Often the analysis of a problem becomes clarified when we switch attention from some space to its dual. This will be an important theme in this course.

Return to dual space. We have chosen a basis $\{e_i\}$ in V. What about a basis in V^*? This would be a set of n linearly independent covectors, call them $\{e^{*i}, i=1,\dots,n\}$, which can be defined by specifying their action on the basis $\{e_i, i=1,\dots,n\}$ in V. The definition

$$e^{*i}(e_j) = \delta^i_j$$

is convenient. It means

$$\alpha = \sum_i \alpha_i \, e^{*i},$$

where $\alpha_i = \alpha(e_i)$ are the components as defined above. Thus those components are the components in the usual sense of α w.r.t. the basis $\{e^{*i}\}$.

The basis $\{e^{*i}\}$ in V^* is said to be <u>dual</u> to the basis $\{e_i\}$ in V.

"Inner product" notation mentioned by Nakara. N. wants to use the notation,

$$\langle \alpha, v \rangle = \alpha(v)$$

and call it an "inner product." This is ok as long as you don't confuse this with the inner product associated with a metric (which we haven't introduced yet). This $\langle\, ,\rangle$ operation is a map,

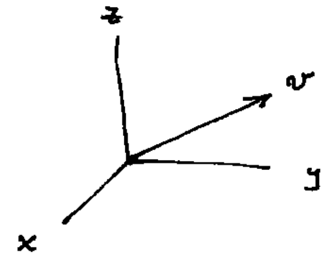
$$\langle\, ,\rangle : V^* \times V \to K : (\alpha, v) \longmapsto \alpha(v).$$

But to be safe I'd prefer not to call this an "inner product" because we will define an (different) inner product later that does involve the metric.

(11)

1/22/04

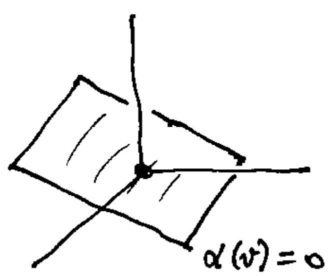
<u>Remark:</u> How to visualize a covector. To visualize a vector $v \in V$ is easy, e.g. if $V = \mathbb{R}^3$,



But what about $\alpha \in V^*$, i.e. $\alpha : V \to \mathbb{R}$ (we'll take $K = \mathbb{R}$ for this discussion). α of course is a vector in V^*, but how can we visualize it in V?

Well, α is a real-valued function on V, so we can look at its contour surfaces (level sets), $\alpha(v) = \text{const}$. The value 0 is particularly interesting, the level set $\alpha(v) = 0$ is otherwise just the kernel of α.

Can easily show that if $\alpha \neq 0$, then $\ker \alpha$ is an $(n-1)$-dimensional vector subspace of V ($\dim V = n$), i.e. a hyperplane passing thru the origin.



And the surfaces $\alpha(v) = \text{const} \neq 0$ are other hyperplanes parallel to this one. So if you specify $\ker \alpha$ (the hyperplane), and the value of α on any parallel (but different) hyperplane, you have specified α. These hyperplanes, especially $\ker \alpha$, are part of the geometrical interpretation of covectors.