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Chapter 2. Mathematical preliminaries. (Begin with terminology about functions, maps, etc.)

Psychology in physics is that a "function" is a real-valued function. Also, in physical applications we often speak of a "many-valued function."

In mathematics, however, a function, map, and mapping are all the same thing, a mapping between sets. If f maps set X into set Y , we write

$$f: X \rightarrow Y$$

\uparrow \nwarrow
 the domain the range.

(Y need not be a set of real numbers, can be any set.)

The function is supposed to be defined for all $x \in X$ and is single-valued, i.e. $y = f(x)$ is a unique element of Y .

Here X, Y are sets and $x \in X, y \in Y$ are elements of those sets. Instead of writing $y = f(x)$ one often writes

$$f: x \mapsto y \quad \left[\text{means same as } y = f(x) \right]$$

Sometimes one writes

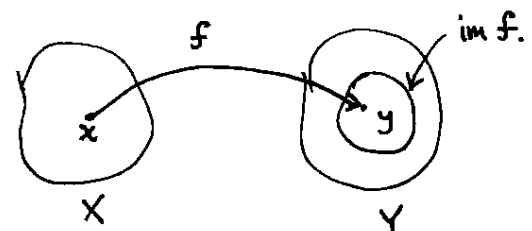
$$f: X \rightarrow Y: x \mapsto y$$

(gives info about domain, range, and names elements $x \in X, y \in Y$ such that $y = f(x)$.)

Although f is defined for all $x \in X$, the y values obtainable as $y = f(x)$ for $x \in X$ do not have to fill up Y (the range).

Define image of f = $\text{im } f \subseteq Y$ by

$$\text{im } f = \{ f(x) \mid x \in X \}.$$

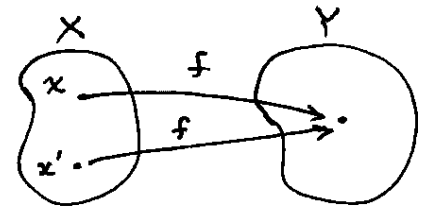
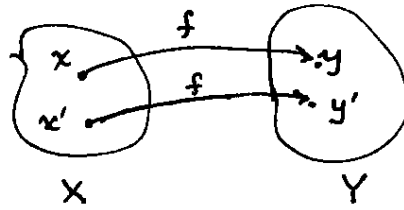


In general, $\text{im } f$ is a (proper) subset of Y .

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Classification of maps.

Injective or one-to-one means that distinct points of X are mapped to distinct points of Y :



can't happen with injective maps.

Injective: if $x \neq x'$, then $f(x) \neq f(x')$.

Surjective or onto means that $\text{im} f = Y$ (f fills up Y).

Bijective means one-to-one and onto.

The inverse function $f^{-1}: Y \rightarrow X$ exists iff f is bijective.

Sometimes we use the notation f^{-1} even when the inverse function doesn't exist. For example, we can interpret

$$f^{-1}(y) = \{x \in X \mid y = f(x)\}$$

this is a set of points in X . = "inverse image of y "

or for $A \subset Y$, $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. "inverse image of A ".

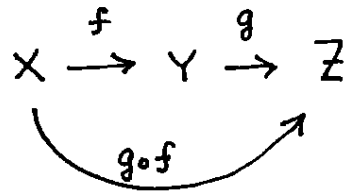
Other map stuff.

Constant map $c: X \rightarrow Y: x \mapsto c(x) = \text{indep. of } x$

Restriction of a map (that is, restriction to a subset of the domain).

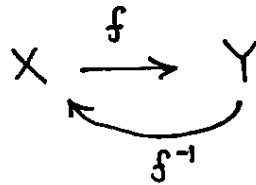
Let $A \subset X$,

$$f|_A: A \rightarrow Y: x \mapsto f(x), \quad x \in A.$$

Composition of maps.

$$(g \circ f)(x) = g(f(x)) \in Z.$$

special case of bijection,



$$f \circ f^{-1} = \text{id}_Y$$

$$f^{-1} \circ f = \text{id}_X$$

id = identity map.

$$\text{id}_X: X \rightarrow X$$

$$\text{id}_Y: Y \rightarrow Y.$$

Inclusion map: $\text{if } A \subset X,$

$$f: A \rightarrow X: a \mapsto a, \quad a \in A.$$

often written, ~~often~~ $A \hookrightarrow X.$

We often have maps between spaces (e.g. groups, vector spaces, ...) that have structure. For example, you can add elements of vector spaces, and multiply elements of groups. If this structure is preserved by the map, then we say it is a homomorphism.

For example, if G, H are groups and $f: G \rightarrow H$, and if

$$f(x_1)f(x_2) = f(x_1x_2) \text{ for all } x_1, x_2 \in G,$$

then f is a group homomorphism.

Another example, if V, W are vector spaces and $f: V \rightarrow W$, and if $f(x_1) + f(x_2) = f(x_1 + x_2)$, then f is a linear homomorphism.

(also known as a linear map or linear operator)

If f is a homomorphism that is also a bijection, then f is said to be an isomorphism, and we write (in this course)

$$X \cong Y \quad (X \text{ and } Y \text{ are } \underline{\text{isomorphic}}).$$

Now we come to the important concepts of equivalence relation, equivalence class, and quotient space, that will be used very frequently in this course.

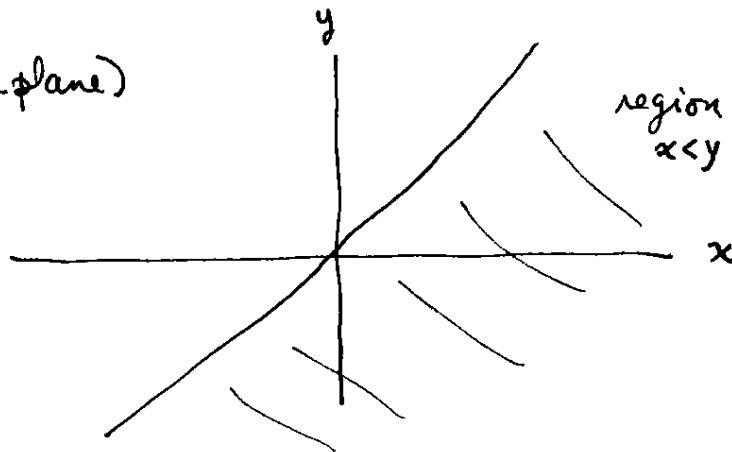
Idea: A relation is a thing like \neq , $=$, $>$, $<$ etc. (on ^{real} numbers).

Def: A relation R on a set X is a subset of $X \times X$,
 \uparrow Cartesian product.
 and we write xRy if $(x,y) \in R$.

Example: Let $X = \mathbb{R}$, $R = <$.

\uparrow real numbers \uparrow the relation (don't confuse them).

Then $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ (the plane)



$x < y$ if (x,y) lies below the diagonal line. Illustration of standard trick in geometry: view something in a higher dimensional space.

Now, an equivalence relation is a relation that has all the formal properties of $=$.

Idea: Objects are equivalent if they are equal insofar as some of their properties are concerned (that is, if we ignore the other properties). For example, suppose we consider two points in \mathbb{R}^3 to be equivalent if they have the same r coordinate (ignore θ, ϕ). This is an equivalence relation.

Def. A relation \sim on a set X is an equivalence relation if

- 1) $a \sim a$ (reflexive)
 - 2) $a \sim b \Rightarrow b \sim a$ (symmetry)
 - 3) $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitive).
- } for all $a, b, c \in X$.

Given an equivalence relation \sim on X , and $x \in X$, we define the

$$\underline{\text{equivalence class of } x} = [x] = \{y \in X \mid y \sim x\}$$

(the set of all elements in X equivalent to x).

In the notation $[x]$, x is called the representative element of the equivalence class. Any other element equivalent to x can be chosen as the representative element, i.e. if $x \sim y$ then $[x] = [y]$.

Basic fact about equivalence relations is that they partition X into mutually disjoint subsets (the equivalence classes). That is, for all $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$ (the empty set).

Proof: Assume $[x] \cap [y] \neq \emptyset$. Then there exists $a \in [x], [y]$ (a common element). This means $a \sim x$ and $a \sim y$, hence $x \sim y$, indeed, every element of $[x] \sim$ every element of $[y]$. Therefore $[x] = [y]$. But either $[x] \cap [y] = \emptyset$ or $[x] \cap [y] \neq \emptyset$. QED

Now define quotient space = set of equivalence classes = $\{[x] | x \in X\}$.

$$= \frac{X}{\sim} \quad (\text{notation}).$$

(this is a new space in which a single point represents an entire equivalence class of X)

Examples of equivalence classes, quotient spaces.

(1). let $X = \mathbb{Z} = \text{integers} = \{\dots, -1, 0, 1, 2, \dots\}$.

Let $n \sim m$ if $n \equiv m \pmod{2}$, i.e. $n-m = \text{even}$.

There are 2 equiv. classes, the even and odd integers. Note, disjoint.

$$\frac{\mathbb{Z}}{\sim} = \{[0], [1]\} = \{\{\text{even \#s}\}, \{\text{odd \#s}\}\} \cong \mathbb{Z}_2$$

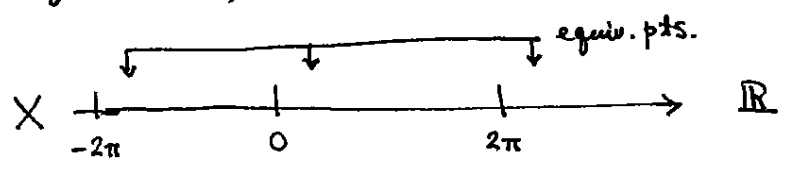
(integers modulo 2).

More generally,

$$\mathbb{Z}_n = \{0, 1, \dots, n-1\} = \text{cyclic group under addition modulo } n.$$

↑
these really mean $[0], [1], \dots$.

(2) Can do something similar with reals. Let $X = \mathbb{R}$, let $x \sim y$ if $x-y = 2\pi n, n \in \mathbb{Z}$.



$$\frac{X}{\sim} = \begin{array}{c} \text{---} \text{---} \text{---} \\ | \quad \quad | \\ 0 \quad \quad 2\pi \end{array} = \{[x] | 0 \leq x < 2\pi\}.$$

This example illustrates how a quotient space may inherit a topology from the larger space from which it was derived.

$$[2\pi] = [0], \quad [2\pi - \epsilon] \text{ near } [0].$$

Thus, $\frac{\mathbb{R}}{\sim} = \text{circle} = S^1$.

(3) States in quantum mechanics. (Pure states)

Suppose not normalized. Then $\langle A \rangle = \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle}$, All physical measurements have this form

invariant under $|\psi\rangle \mapsto c|\psi\rangle$ where $c \in \mathbb{C}, c \neq 0$.

Defines equivalence class of kets.

So physical state is specified, not so much by $|\psi\rangle$, but

by equivalence class of kets $[|\psi\rangle] = \{c|\psi\rangle \mid c \in \mathbb{C}, c \neq 0\}$.

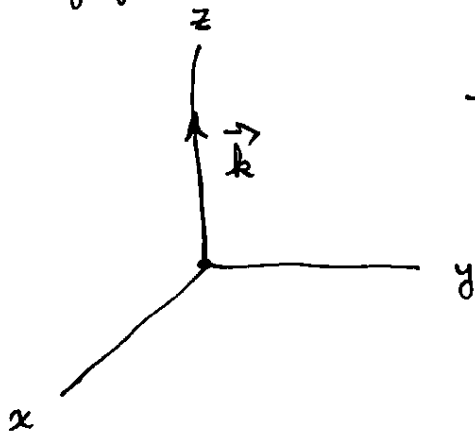
← (the quotient space).

This is example of complex projective space (more on that later).

$[|\psi\rangle]$ is called a ray in the Hilbert space.

(4) Wave functions $\psi(x)$ in quantum mechanics. You can't measure $\psi(x)$ (at a given point). (More in class.)

(5) Polarizations in classical EM theory. Consider a classical light wave of fixed ω (hence $k = \omega/c$) propagating in the z -direction. vacuum



$$\vec{E} = \text{Re} \left[\vec{a} e^{i(kz - \omega t)} \right]$$

↳ complex amplitude,

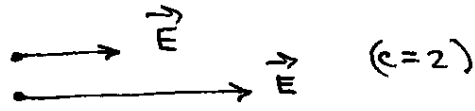
$$\vec{a} = a_x \hat{x} + a_y \hat{y},$$

$$a_x, a_y \in \mathbb{C}.$$

Wave is specified by \vec{a} , hence space of waves is $\mathbb{C}^2 = \mathbb{R}^4$

But if you want polarization to be meaningful, exclude $\vec{a} = 0$, hence space of waves we consider is ~~\mathbb{C}^2~~ $\mathbb{C}^2 \setminus \{0\} = \mathbb{R}^4 \setminus \{0\}$.

Now polarization is supposed to involve only the direction of the \vec{E} field, not its magnitude. So \vec{a} and $c\vec{a}$, $c \in \mathbb{R}$, $c > 0$, represent the same polarization state.



This defines an equivalence class, $\frac{\mathbb{R}^4 \setminus \{0\}}{\sim (\text{ampl})} = S^3$,
 quotient space is 3D sphere in \mathbb{R}^4 ,

But the state of polarization of a light wave is also independent of the phase of the wave (equivalent to choice of origin of time). This means that

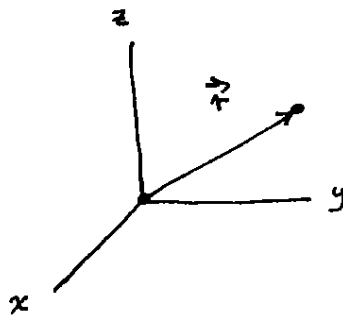
$$\vec{a} \text{ and } e^{i\alpha} \vec{a}, \quad \alpha = \text{real},$$

have the same polarization. Defines new \sim (now have 2 \sim s, one for amplitude and one for phase). Turns out

$$\frac{S^3}{\sim (\text{phase})} = S^2. \quad \left(\begin{array}{l} \swarrow \text{not completely obvious} \\ \leftarrow \text{this is the Hopf fibration, more on that later.} \end{array} \right)$$

The space of polarization states is S^2 , sometimes called the Poincaré sphere (Stokes' parameters are coordinates on S^2).

(6) Central force motion.



let $\vec{r}_1 \sim \vec{r}_2$ if $|\vec{r}_1| = |\vec{r}_2|$. Equivalence class = sphere S^2 .

$$\text{Quotient space} = \frac{\mathbb{R}^3}{\sim} = \{r \in \mathbb{R} \mid r \geq 0\} = \begin{array}{c} \longleftarrow r \\ \bullet \\ 0 \end{array} = \text{radial half-line}$$

= space on which radial wave functions live in Q.M.