Physics 222
Spring 2004
Homework and Notes 9
Due 5pm, Friday, April 16, 2004

Reading Assignment: Nakahara, pp. 251–273. See also Frankel, pp. 232–246.

Notes. Just a few comments. On pp. 258–259, Nakahara is trying to point out that if the connection coefficients $\Gamma^\mu_{\alpha\beta}$ vanish in some coordinate system, then it means that the rule of parallel transport (in those coordinates) is that the parallel transported vector has the same components as the original vector. This applies, for example, to parallel transport in a vector space, using linear coordinates. One can similarly define such a (trivial) connection on any parallelizable manifold, which includes group manifolds, as pointed out in class. Obviously, if the connection coefficients vanish (in some coordinates), then the curvature tensor $R^\mu{}_{\nu\alpha\beta}$ vanishes (in all coordinates).

On p. 253, Nakahara says that the variational condition gives the local extremum of the length of a curve between two points. Actually, it is only a stationary point, in general (a kind of a saddle point in function space).

On p. 265, Nakahara's equation (7.68b) is wrong, but he never uses it in the subsequent derivation. Just below that, I can't see what dividing by x' has to do with anything. Actually, the derivation of the geodesic equations are much easier if you just use a Lagrangian. In the present case, let

$$L(x,y,x',y') = \frac{1}{2}\frac{x'^2+y'^2}{y^2}, \tag{1}$$

where the 1/2 is only for convenience. (Also, we could have used the square root of the above expression, but the answers will be the same and the above expression is easier to work with.) You will find the Euler-Lagrange equations give you the geodesic equations immediately, and also (by Noether's theorem on the ignorable coordinate x) they give you the integral (7.69).

1. (DTB) Given (M,∇,g) (manifold with connection plus metric), and assume $\nabla g = 0$, but don't make any other assumptions. In particular, don't assume that the torsion $T=0$. Prove that

$$g(W,R(X,Y)Z) + g(Z,R(X,Y)W) = 0, \tag{2}$$

where X, Y, Z, W are vector fields. Do this in coordinate-free notation. What symmetry does this imply for $R^\mu{}_{\nu\alpha\beta}$?

2. Consider an accelerated particle moving in Minkowski space-time, with metric

$$g = dt^2 - dx^2 - dy^2 - dz^2. \tag{3}$$

The particle's world line is $x^\mu(\tau)$, where τ is proper time, and its spin 4-vector is s^μ. The spin is required to be a purely spatial vector in the particle's rest frame, so $s^\mu u_\mu = 0$, where $u^\mu = dx^\mu/d\tau$. In the absence of torques (e.g., magnetic fields) on the spin, the spin vector evolves according to the equation of Fermi-Walker transport,

$$\frac{ds^\mu}{d\tau} = -u^\mu(s^\nu a_\nu), \tag{4}$$

where $a^\mu = du^\mu/d\tau$. This equation guarantees that s^μ is orthogonal to u^μ for all τ if it is orthogonal at $\tau = 0$ (as we assume).

If the velocity of the particle undergoes a cycle in some time interval, so that the final velocity is equal to the initial velocity, then the final 3-dimensional space-like hyperplane that the spin lies in is the same at the beginning and ending of the the time interval (say, τ_0 to τ_1). (The particle might be undergoing periodic motion.) Under these circumstances, it is meaningful to talk about the 3-dimensional rotation (in the 3-dimensional, space-like hyperplane) that maps the initial spin into the final spin. This rotation is usually called Thomas precession.

The world velocity of the particle u^μ satisfies

$$u^\mu u_\mu = 1. \tag{5}$$

Thus, the vector u^μ lies on a hyperboloid in "world velocity space". Call the hyperboloid H. "World velocity space" has the same metric $g_{\mu\nu}$ as Minkowski space. We create a map from the world line of the particle to H by means of the function $u^\mu(\tau)$, which defines a curve on H. Call this the "H-map." If the velocity is cyclic, as assumed, then the curve on H forms a loop with a base point $u^\mu(\tau_0) = u^\mu(\tau_1)$.

As pointed out in class, the surface H is actually a purely space-like hypersurface of constant negative curvature, if its metric is taken to be the (Minkowski) metric of the imbedding space, restricted to H. Explain why the rotation of Thomas precession is the holonomy of the loop on H, with respect to this metric on H.

This can largely be done in words and pictures, especially if you have done problem 3 of last week.

In this problem we are dealing with what is called the "normal bundle", that is, the set of vectors normal to a submanifold of a (pseudo)-Riemannian manifold. In this case, the submanifold is the 1-dimensional world line of the particle. The rate of rotation of the spin along the world line is the angular velocity of Thomas precession, essentially an $\mathfrak{so}(3)$-valued 1-form on the world line, which is the pull-back of the Levi-Civita connection on H to the world line by means of the H-map.

3. (DTB) The geometrical setting that is appropriate for classical Hamiltonian mechanics is a *symplectic manifold*. This is a manifold P (for "phase space") endowed with a nondegenerate, closed 2-form ω. To say that it is non-degenerate means that the component matrix $\omega_{\mu\nu}$ is nonsingular in any basis. In addition, $d\omega = 0$. Since ω is nondegenerate, the matrix inverse to $\omega_{\mu\nu}$ exists; call it

$\omega^{\mu\nu}$, so that

$$\omega_{\mu\sigma}\omega^{\sigma\nu} = \delta^\nu_\mu. \tag{6}$$

We use $\omega_{\mu\nu}$ and $\omega^{\mu\nu}$ to raise and lower indices on P.

(a) Show that the dimension of a symplectic manifold is always even, $2N$. N is the *number of degrees of freedom.*

(b) A *classical observable* A is a scalar field on a symplectic manifold, $A : P \to \mathbb{R}$. A particular observable of interest is the Hamiltonian H. It is associated with a vector field X, the equations of motion of the classical system, by *Hamilton's equations*,

$$i_X\omega = -dH, \tag{7}$$

or

$$X^\mu = \omega^{\mu\nu} H_{,\nu}. \tag{8}$$

When we wish to emphasize that X is associated with Hamiltonian H, we will write X_H.

The *Poisson bracket* of two classical observables A and B is defined by

$$\{A, B\} = A_{,\mu}\, \omega^{\mu\nu}\, B_{,\nu}. \tag{9}$$

Show that the Poisson bracket satisfies the Jacobi identity,

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{10}$$

Also, find a relationship between the Lie bracket $[X_A, X_B]$ of two Hamiltonian vector fields and the Poisson bracket $\{A, B\}$.

(c) In the subject known as "deformation quantization," the idea is to deform the usual (pointwise, commutative) multiplcation law for classical observables, by introducing an $\hbar$ dependence into the multiplication law. When $\hbar = 0$, we have the classical algebra of observables, but when $\hbar$ is switched on, the multiplication law becomes noncommutative (but it is still associative). The new multiplication law is interpreted as that of quantum observables.

In deformation quantization, it is necessary to introduce a "symplectic connection," that is, a connection ∇ that preserves the symplectic 2-form, $\nabla\omega = 0$. It is usually assumed that this connection has vanishing torsion. In the analogous circumstances in metrical geometry, it is possible to solve for the connection coefficients $\Gamma^\mu_{\alpha\beta}$ in terms of the metric tensor $g_{\mu\nu}$ and its derivatives (this is the Levi-Civita connection). Can this also be done for a symplectic connection? Show that if Γ and $\bar{\Gamma}$ are the components of two torsion-free symplectic connections, then the difference

$$\Gamma_{\mu\alpha\beta} - \bar{\Gamma}_{\mu\alpha\beta} \tag{11}$$

is a completely symmetric tensor (here $\Gamma_{\mu\alpha\beta} = \omega_{\mu\nu}\Gamma^\nu_{\alpha\beta}$).