Physics 222
Spring 2004
Homework and Notes 8
Due 5pm, Friday, April 9, 2004

Reading Assignment: Nakahara, pp. 228–250. See also Frankel, pp. 110-122, 155–162, 241–244.

Notes. Nakahara's approach to integration uses singular simplexes, whereas I used singular cubes in lecture. Since a cube can be divided into simplexes or mapped into a simplex, it makes no difference for the development of the theory which is used. I chose cubes because it makes the proof of Stokes' theorem a little easier. In general, you may think of a chain as a linear combination of "singular" objects, that is, mappings from some region in $\mathbb{R}^r$ to M.

On p. 233, Nakahara says that d is the adjoint of ∂. This is incorrect, according to his definition of the adjoint, which requires a metric. It is correct to say, however, that d is the pull-back of ∂. More exactly, $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ is the pull-back of $\partial_{r+1} : C_{r+1}(M) \to C_r(M)$, as looks reasonable if we identify $C_r(M)^*$ with $\Omega^r(M)$. The pull-back does not require a metric.

In Corollary 6.1 on p. 234, Nakahara means to say that the set of cohomology classes $\{[c_i]\}$ are linearly independent (just $[c_i] \neq [c_j]$ is not enough).

Nakahara's proof of the Poincaré lemma (pp. 235–237) is straightforward to follow, but not very illuminating. A more geometrical approach to the subject was given in lecture. Notice that Nakahara's statement of Theorem 6.3 should say, "... any closed r-form on U for $r \geq 1$ is also exact." That is, the theorem as stated is not true for $r = 0$. Another way to state this version of the Poincaré lemma is to say that on a contractible region R, $H^r(R)$ is the same as $H^r(p_0)$, where p_0 is the contraction point.

For Poincaré duality (p. 238), just note the result, Eq. (6.37). It is impossible to understand the logic, based on material we've covered so far.

Nakahara's theorem 6.5, p. 241, is trivial, if you note that $\pi_1(M) = \{0\}$ implies $H_1(M) = \{0\}$ implies $H^1(M) = \{0\}$.

Nakahara's Eq. (7.3) is really ugly, in that it mixes the old and new meanings of dx in one formula. (Old, dx is a small number; new, dx is an operator on vectors.)

1. Let M be a manifold of dimension D. Suppose there is a deformation retract of M onto a submanifold of dimension $d < D$. For example, you may think of the solid interior of a 2-torus (imbedded in $\mathbb{R}^3$), being contracted onto a circle. Suppose the deformation retract can be achieved as a result of an advance map Φ_t associated with some vector field $X \in \mathfrak{X}(M)$, as t goes from 0 to T. (This assumption is not necessary, it just makes the problem a little easier.) Show that $H^r(M) = \{0\}$ for $r > d$.

2. In class we derived the formula,

$$\int_0^T dt \int_c \Phi_t^* i_X d\omega = \int_c \Phi_T^* \omega - \int_c \omega - \int_0^T dt \int_c d\Phi_t^* i_X \omega, \tag{1}$$

where c is an r-chain on M, ω is an r-form on M, X is a vector field on M and Φ_t is the associated advance map. Let $m = \dim M$. On a region of M diffeomorphic to the m-ball (the region $r \leq 1$ in $\mathbb{R}^m$), let X be the flow that contracts the ball to its center,

$$X^\mu(x) = -x^\mu. \tag{2}$$

Use this flow to derive the Volterra formula for the potential of a closed form ω.

3. Let M be a submanifold of Euclidean $\mathbb{R}^n$. The metric on $\mathbb{R}^n$ is

$$g = \sum_{i=1}^n dx^i \otimes dx^i, \tag{3}$$

where $\{x^i, i = 1, \ldots, n\}$ are the standard coordinates on $\mathbb{R}^n$. Let $\{x^\mu, \mu = 1, \ldots, m\}$ (with $m \leq n$) be coordinates on M, which is specified by functions $x^i = x^i(x^\mu)$. Let the metric on the submanifold be the metric on $\mathbb{R}^n$, restricted to the submanifold. The metric on M has components $g_{\mu\nu}$. Let x^μ and $x^\mu + \xi^\mu$ be coordinates of two nearby points (call them x and $x + \xi$) on M (ξ^μ is infinitesimal). As discussed in class, we define a connection on M as follows. We take a tangent vector X in T_xM, reinterpret it as a tangent vector in $T_x\mathbb{R}^n$, parallel transport it over to $T_{x+\xi}\mathbb{R}^n$ by using the vector space structure of $\mathbb{R}^n$, then project it onto $T_{x+\xi}M$ using the metric in $\mathbb{R}^n$. Find the connection coefficients $\Gamma^\mu_{\alpha\beta}$ in terms of $g_{\mu\nu}$.