Physics 222
Spring 2004
Homework and Notes 6
Due 5pm, Friday, March 12, 2004

Reading Assignment: Nakahara, pp. 191–204, 207–208, 216–222. See also Frankel, 58–80, 89–94, 125–136.

Notes. Equation (5.83), p. 202, is meaningless. Please ignore it. Otherwise the material on pp. 191–204 is ok. I skipped the classical mechanics of Hamiltonian vector fields in class, but it is an interesting application of differential forms. Classical mechanics is geometrical at heart (see below). I also skipped the material on integration of forms in lecture, but we will come back to it. Next week we will work through some basics of the differential geometry of Lie groups, and then do integration of differential forms.

1. (DTB) The *Kronecker tensor* δ is a type $(1,1)$ tensor. Make sure you understand the difference between a tensor at a point $x \in M$ and a tensor field. δ at a point $x \in M$ is a map,

$$\delta|_x : T_x^* M \times T_x M \to \mathbb{R}, \tag{6.1}$$

whereas as a field δ is a map,

$$\delta : \mathfrak{X}^*(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M) : (\alpha, Y) \mapsto \alpha(Y), \tag{6.2}$$

where the final formula defines δ. Let $X \in \mathfrak{X}(M)$, and compute $L_X \delta$ (the Lie derivative of δ along X).

2. (DTB) Nakahara Exercise 5.15, p. 199 (Exercise 5.32, p. 161 of the first edition).

3. (DTB) As was discussed in class, the Lie derivative L_X obeys the Leibnitz rule when acting on tensor products. As was also discussed, the exterior product $\wedge$ is an antisymmetrized tensor product.

(a) Let $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^s(M)$. Find and expression for $L_X(\alpha \wedge \beta)$ in terms of $L_X \alpha$ and $L_X \beta$.

(b) The *Cartan formula* is

$$L_X = i_X d + d i_X, \tag{6.3}$$

where $X \in \mathfrak{X}(M)$, valid when both sides act on differential forms. Show that the right hand side obeys the same rule when acting on $\alpha \wedge \beta$ as does L_X in part (a).

(c) Show that the Cartan formula (6.3) is valid when acting on 0-forms and 1-forms.

(d) Explain why parts (a)–(c) prove the Cartan formula in all cases (that is, when acting on arbitrary differential forms).

4. A problem on coordinate and noncoordinate bases. Only part (a) is marked DTB.

(a) (DTB) Let x^μ be the coordinates in a chart on manifold M. Inside the domain of the chart, the vector fields $\{\partial/\partial x^\mu\}$ form a basis in the tangent spaces to M, that is, if these vector fields are evaluated at a point $p \in M$, then they form a set of $m = \dim M$ linearly independent vectors in T_pM, for each p in the domain of the chart. These vector fields are said to form a *coordinate basis*.

Let $\{e_\mu, \mu = 1, \ldots, m\}$ be a new set of vector fields that are also linearly independent at each point p in some region contained in the domain of the chart x^μ. Notice that the μ subscript on e_μ is not a component index, rather it serves to label the vector fields. The e_μ can be expanded as linear combinations of the coordinate basis vectors,

$$e_\mu = e_\mu^{\ \nu} \frac{\partial}{\partial x^\nu}, \tag{6.4}$$

where $e_\mu^{\ \nu}$ are the expansion coefficients. These are functions of position in M.

Show that the set $\{e_\mu\}$ is also a coordinate basis, that is, that there exist scalars y^μ such that

$$e_\mu = \frac{\partial}{\partial y^\mu}, \tag{6.5}$$

if and only if

$$[e_\mu, e_\nu] = 0, \tag{6.6}$$

where [,] is the Lie bracket. This is a local construction.

Whether or not the set $\{e_\mu\}$ is a coordinate basis, the Lie brackets of the basis fields among themselves are interesting. These Lie brackets are themselves vector fields, and so can be expanded as linear combinations of the basis vectors. That is, an expansion of the form,

$$[e_\mu, e_\nu] = c_{\mu\nu}^{\ \ \sigma}\, e_\sigma \tag{6.7}$$

exists. The expansion coefficients $c_{\mu\nu}^{\ \ \sigma}$ are called the *structure constants*, although in this context they are not constant (they are functions of position).

(b) Let M be the configuration space of a mechanical system in classical mechanics, with coordinates x^μ imposed (in some chart). These are what are called "generalized coordinates" in classical mechanics, meaning that they are not necessarily Cartesian coordinates (nor for that matter is M necessarily a vector space $\mathbb{R}^m$).

A configuration point at position $x \in M$ may be given any velocity. The combination of the position and velocity of the configuration constitutes what we call the *dynamical state* of the system, because the knowledge of the dynamical state allows us to determine the subsequent evolution. The velocity is specified by the time derivatives $\dot{x}^\mu$, which can be regarded as the components in the coordinate basis of the velocity vector $V \in T_x M$,

$$V = \dot{x}^\mu \frac{\partial}{\partial x^\mu}. \tag{6.8}$$

The *state space* of the system is the space of all possible dynamical states, that is, all possible configurations with all possible velocities at a given configuration. This space is the tangent bundle TM. (See the lecture notes for the definition of TM.)

The Lagrangian is usually regarded as a function $L(x^\mu, \dot{x}^\mu)$, but we can see it abstractly as a scalar $L : TM \to \mathbb{R}$. The equations of motion (the Euler-Lagrange equations) are

$$\frac{dp_\mu}{dt} = \frac{\partial L}{\partial x^\mu}, \tag{6.9}$$

where p_μ is the *canonical momentum*, defined by

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}. \tag{6.10}$$

Suppose we write the tangent vector V as a linear combination of some other basis $\{e_\mu\}$ (besides the coordinate basis $\{\partial/\partial x^\mu\}$),

$$V = \dot{x}^\mu \frac{\partial}{\partial x^\mu} = v^\mu \, e_\mu, \tag{6.11}$$

which defines the components v^μ with respect to the new basis (itself not necessarily a coordinate basis). Then we can transform the Lagrangian from the variables $\dot{x}^\mu$ to the variables v^μ. We write this tranformation,

$$L(x^\mu, \dot{x}^\mu) = \bar{L}(x^\mu, v^\mu), \tag{6.12}$$

using a new symbol $\bar{L}$ to indicate that the Lagrangian has been expressed in terms of new variables. Also, let us define

$$\pi_\mu = \frac{\partial \bar{L}}{\partial v^\mu}, \tag{6.13}$$

which as it turns out are the components of the momentum with respect to the basis dual to $\{e_\mu\}$.

Find a nice equation of evolution for $d\pi_\mu/dt$ in terms of the derivatives of the Lagrangian $\bar{L}$ and the structure constants of the basis.