Physics 222
Spring 2004
Homework and Notes 5
Due 5pm, Friday, March 5, 2004

Reading Assignment: Nakahara, pp. 135–153 and 169–191. I also recommend Frankel, pp. 3–56.

Notes. Here are some comments on the text.

The notation in Eq. 4.14, p. 134 (Eq. 4.10 on p. 102 of the first edition) is a little misleading, since he wrote the subset X of G in the form $\{x_j\}$. The point is that x_1, x_2, etc., in Eq. 4.14 refer to any elements of X taken in any order, not necessarily the first, second, etc., elements of X. Some of the x_i's in Eq. 4.10 are even allowed to be identical (but not adjacent ones). On the last line of p. 102 (first edition), he should write i_j instead of n_j (this is corrected in the second edition).

In the first sentence of Sec. 4.9.4, Nakahara means $\pi_3(\mathbb{R}P^2)$ (subscript 3 omitted).

1. (DTB) A problem on covering spaces. If $\bar{M}$ is a covering space of a connected space M, then $\bar{M}$ covers M a certain number of times. For example, $SU(2)$ is the double cover of $SO(3)$. In this problem you will show that the number of times $\bar{M}$ covers M (if finite) is a divisor of the order of $\pi_1(M)$. (The *order* of a group is the number of elements in it). Thus, the maximum number of times that $\bar{M}$ can cover M is the order of $\pi_1(M)$ (in this case, $\bar{M}$ is the universal covering space). I'm marking this whole problem as DTB, but parts (a) and (b) are standard material in introductory group theory, while parts (c) and (d) are rather different material. So you may want to claim DTB for only half of this problem.

(a) Review homework problem 1.1 on group actions. Let G be a group of finite order, and let H be a subgroup. Let G/H be the space of cosets (left or right, it won't matter). H is not necessarily a normal subgroup, so G/H is not necessarily a group. Let $\#S$ stand for the number of elements in any set S. Thus, $\#G$ is the order of G. Show that every coset of H in G has the same number of elements, namely, $\#H$. Conclude, therefore, that

$$\#\Big(\frac{G}{H}\Big) = \frac{\#G}{\#H}, \tag{5.1}$$

where $\#(G/H)$ is the number of cosets. Thus, the order of H must be a divisor of the order of G. (For example, a group with 6 elements can have subgroups of order 2 or 3, but not 4 or 5.)

(b) Let $g \mapsto \Phi_g$ be an action of a group G on a space X. Let the orbits of the action be labelled by representative points in them, for example $[x]$ is the set $\{\Phi_g x | g \in G\}$. For given $x \in X$, let

$$I_x = \{g \in G | \Phi_g x = x\}. \tag{5.2}$$

Show that I_x is a subgroup of G. I_x is called the *isotropy subgroup* or *stabilizer* of G at $x \in X$. (Different points of X may have different stabilizers.) Show that points in $[x]$ can be uniquely labelled by the cosets in G/I_x. Conclude, therefore, that the number of distinct elements in an orbit is $\#G/\#I_x$.

(c) Let M be a connected topological space. Let $\bar{M}$ be another space that satisfies all the conditions of a covering space (see p. 149 of Nakahara), except that we will not require that $\bar{M}$ be connected. We let $p : \bar{M} \to M$ be the projection map, as in the text. Let x_0 be a chosen point of M, and let $F_0 = p^{-1}(x_0)$. Notice that by the definition, F_0 consists of a discrete set of points in $\bar{M}$. Let $G = \pi_1(M, x_0)$. Given a loop α based at x_0 in M and a point $\bar{x}_0 \in F_0$, you can follow the lift of α as you go around α, and you will return at another point of F_0. This specifies a map $\Phi : F_0 \to F_0$.

[Here we are invoking the theorem discussed in class, that given a path $\alpha : [0, 1] \to M$ such that $\alpha(0) = x_0$, and given a point $\bar{x}_0 \in p^{-1}(x_0)$, there exists a unique smooth curve in $\bar{M}$, $\bar{\alpha} : [0, 1] \to \bar{M}$, such that $\bar{\alpha}(0) = \bar{x}_0$ and such that $p(\bar{\alpha}(t)) = \alpha(t)$.]

The map Φ depends on the loop α, but show that it is the same map for any loop α' that is homotopic to α. Thus, Φ only depends on the equivalence class $[\alpha]$, which otherwise is an element of $G = \pi_1(M)$. Write $g = [\alpha]$, and write Φ_g for the map previously denoted simply Φ, to indicate the class of loops it depends on. Find the multiplication law for $\Phi_{g_1}\Phi_{g_2}$. Is $g \mapsto \Phi_g$ an action of G on F_0? If not, find an action of G on F_0. (Note that $\alpha * \beta$ means, follow α first, then β.)

(d) Show that $\bar{M}$ is connected iff F_0 consists of a single orbit of the action you have constructed. Hence conclude, in the case that $\bar{M}$ is connected and $\#G$ is finite, that $\#F_0$ is a divisor of $\#G$.

2. Nakahara, Exercises 5.3 and 5.5, p. 187 (Exercises 5.13, 5.15, pp. 148–149 of the first edition).

3. In this problem we consider the behavior of vector fields and advance maps under diffeomorphisms. The background is that we are given a vector field X on a manifold M, and a diffeomorphism $f : M \to N$. As explained in class, since f is a diffeomorphism, $f_* X$ is a vector field on N ($X \in \mathfrak{X}(M)$ and $f_* X \in \mathfrak{X}(N)$). We wish to show that advance maps commute with the action of f, that is,

$$f \circ \Phi_t = \Psi_t \circ f, \tag{5.3}$$

where $\Phi_t : M \to M$ is the advance map for X, and $\Psi_t : N \to N$ is the advance map for $f_* X$.

Let $\sigma : [a, b] \to M$ be an integral curve of X passing through $x_0 \in M$ at $t = 0$, and let $\tau : [a, b] \to N$ be given by $\tau = f \circ \sigma$. Show that τ is an integral curve of $f_* X$ passing through $y_0 = f(x_0)$ at $t = 0$. Do this in some local coordinates (say x^i on M, and y^i on N). This proves Eq. (5.3). This is an example of a proof that is easy in coordinates, but more taxing when put into coordinate-free language.