Physics 222
Spring 2004
Homework and Notes 12
Due 5pm, Wednesday, May 12, 2004

Note: I am making this homework due one day after the last day of class. I will hold office hours 1–2pm on Wednesday, May 12.

Reading Assignment: Nakahara, pp. 348–373, and as far as we get into Chapter 10.

Notes. In lecture I did (or will) cover most of the material in Nakahara's Chapter 9, but I tried to make it more clear and motivated. So you might find it most useful to read the lecture notes first, then Nakahara's Chapter 9. This chapter does what can be done with fiber bundles before you introduce a connection, which is the subject of Chapter 10.

On p. 351, Nakahara requires the transition functions t_{ij} to satisfy conditions (9.6a–c), but these conditions follow immediately from the definitions of the t_{ij} (his Eq. (9.4)). (You do not have to impose any extra requirements.)

On p. 352, below Eq. (9.8), Nakahara says that the g_i should be homeomorphisms. For all our applications, they should be diffeomorphisms. In the following paragraph, I think Nakahara means $\Gamma(M, E)$ for the set of sections over M (he has F instead of E).

On the last line of p. 353, Nakahara writes $f' = t_{ij} f$, but he should swap f and f'. This error was also in the first edition.

In Sec. 9.2.3 on bundle maps, a little thought will show that just preserving fibers is not a strong enough condition on a map between fiber bundles that is supposed to define a kind of a bundle homomorphism. (It's obvious that a proper definition must also involve the group and the transition functions.) The proper definition is given in Steenrod. I never used this concept in lecture.

In class I shall present an alternative definition of the pullback of a bundle, from that used by Nakahara in Sec. 9.2.5. His definition is clever geometrically, but I think not very motivated. I will make it a special case of the reconstruction of a bundle.

Nakahara's statement at the bottom of p. 356, about π_2 having maximal rank m, makes no sense. π_2 maps a $2m$-dimensional manifold to another one, so the maximal rank is $2m$. I do not know what he is trying to say in this section. If anyone can figure it out, please let me know.

On p. 357, Nakahara never actually says what the "homotopy axiom" is. But the theorem that a bundle is trivial if the base space is contractible is an important one, and worthy of a special name.

I shall skip the material on the canonical line bundle, since we skipped Chapter 8.

On p. 359, in the first paragraph of Sec. 9.3.2 (on Frames), Nakahara seems to be saying that the components of the basis vectors are "unit vectors" in their own basis. This is a trivial statement of linear algebra and rather pointless in this context.

I will probably skip the material on Whitney sum bundles and tensor product bundles in lecture, due to lack of time.

On p. 363, the fact that the structure group has a natural action on every principal fiber bundle is important. I gave a couple of motivating examples in lecture because the general statement of this fact (Nakahara's Eq. (9.41), essentially reproduced in lecture) is (to me) rather unmotivated.

On p. 364, second paragraph, he means $s_i(p)$ instead of $s_1(p)$. If you have a principal fiber bundle, then, as pointed out in class, the fibers are diffeomorphic to the structure group, but they are not groups since they have no preferred origin. But if you choose an origin by some prescription, then you automatically get a specific identification of the fiber with the structure group. This is what a local section does: it picks out one point on each fiber (over the local U_i), which serves as an "origin" for the fiber. Conversely, if you use the local trivialization ϕ_i, with constant group element (say, $g = e$), then this maps U_i onto P creating a local section. This is what Nakahara calls the "canonical local section." I'd prefer not to call it "canonical", since anything that depends on the local trivialization is highly arbitrary.

In Example 9.7, Nakahara never actually explains why the $U(1)$ bundle over S^2 is relevant for the Dirac monopole. This will be explained in class. On p. 365, below Eq. (9.44), he says, "Take a transition function $t_{NS}(p)$ of the form . . ., so that $t_{NS}(p)$ may be uniquely defined on the equator." What he means is that t_{NS} must be periodic on the equator, since it is supposed to be a smooth function. But there is no reason why it should be $e^{in\phi}$. As explained in class, the real issue is that a local gauge transformation can change $t_{NS}(\phi)$ into any other function belonging to the same homotopy class of $\pi_1(S^1)$, but it cannot change that homotopy class. The function $t_{NS}(\phi) = e^{in\phi}$ belongs to homotopy class n, so every given $t_{NS}(\phi)$ can be gauged into $e^{in\phi}$ for some n.

Nakahara's definition of the Hopf map would look simpler to the eye of a physicist if he changed the sign on Eq. (9.53b) and wrote the result in the form, $\boldsymbol{\xi} = \langle z|\boldsymbol{\sigma}|z\rangle$, as discussed in class. As we say, $\boldsymbol{\xi}$ is the "direction" the spinor z is "pointing in." This is common language in quantum mechanics, but it only works for a spin 1/2 particle (for other spins, the expectation value of the spin operator is not in general a unit vector, and it does not specify the spinor, even to within a phase).

Notice that an example of Eq. (9.64) (for $n = 3$) was proved in a homework. The proof for general n is similar.

On p. 370, Nakahara gives a definition of a fiber bundle associated with a principal fiber bundle as $(P \times F)/G$ (the action of G on $P \times F$ is given by his Eq. (9.66)). In class I explained associated bundles in terms of the reconstruction program. I think I know why Nakahara does it his way, it leads to a useful point of view in Kaluza-Klein theories, for example. But for now I did not want to multiply constructions when one would do.

On p. 371, Nakahara mentions the obstruction to the construction of the spin bundle. This concerns the possibility of defining Dirac or Weyl spinors over a topologically nontrivial space-time (for example, with worm holes). It involves the cohomology classes of the space-time manifold, not over $\mathbb{R}$ but over $\mathbb{Z}_2$. This topic is given a nice discussion by Frankel.

1. In class we showed that S^1 bundles over S^2 are characterized by homotopy classes in $\pi_1(S^1)$. That is, the transition function t_{NS}, which maps the equator onto the S^1, lies in a homotopy class which characterizes the bundle topologically. We found the Hopf bundle $\pi : S^3 \to S^2$ has homotopy class -1 (or $+1$ if you use Nakahara's definition of π).

In homework 1, problem 2, you worked out the homogeneous space $SO(3)/SO(2)$, and found that it was S^2. As pointed out in class, the foliation of a Lie group G into cosets by a Lie subgroup H always endows G with the structure of a principal fiber bundle, in which H is the structure group. This is one circle bundle over a sphere. The orthogonal frame bundle on S^2 with the usual metric is another circle bundle over S^2, also a principal fiber bundle. Find the homotopy classes of these two bundles.

2. Nakahara problem 9.2, p. 372 (problem 2, p. 328, of the first edition).

3. A problem on the Frobenius theorem. As discussed in class, a k-*distribution* over a manifold M ($\dim M = m$) is an assignment of k-dimensional subspaces ($k \le m$) in the tangent spaces T_xM for all $x \in M$ (or perhaps only over some region $U \in M$). The assignment is assumed to be smooth. We say that a k-distribution (in some region) is *integrable* if there exists an $(m-k)$-parameter family of k-dimensional surfaces in the region which are everywhere tangent to the distribution. The surfaces are only required to exist locally.

Let X, Y be vector fields on M which lie in a k-distribution Δ. If Δ is integrable, then the integral curves of both X and Y must lie in the surfaces tangent to Δ. Therefore if we follow the integral curves of X or Y in any order for any elapsed parameters, we must always remain on a given surface. Considering infinitesimal elapsed parameters, we see that the Lie bracket $[X, Y]$ must be tangent to the surface, that is, it must lie in the distribution. Thus, we have a theorem, that if Δ is integrable and $X, Y \in \Delta$, then $[X, Y] \in \Delta$. It turns out the converse is also true (at least locally): if for every vector fields $X, Y \in \Delta$, we have $[X, Y] \in \Delta$, then Δ is (locally) integrable.

Let θ^α, $\alpha = 1, \ldots, m-k$ be a set of linearly independent 1-forms which annihilate Δ. Show that the following four conditions are equivalent:

(i) For every $X, Y \in \Delta$, $[X, Y] \in \Delta$.

(ii) $d\theta^\alpha(X, Y) = 0$ for all vector fields $X, Y \in \Delta$.

(iii) There exist 1-forms $\lambda^\alpha{}_\beta$ such that $d\theta^\alpha = \lambda^\alpha{}_\beta \wedge \theta^\beta$.

(iv) $d\theta^\alpha \wedge \Omega = 0$, where $\Omega = \theta^1 \wedge \ldots \wedge \theta^{m-k}$.