Physics 222
Spring 2004
Homework and Notes 1
Due Friday, January 30, 2004

About Homework: I will try to have weekly homework, but there may be some weeks without it. Nakahara's problems are usually not very good, so I will try to do better. The homework will be made available on the web site by Friday of each week, and will be due at 5pm on Friday of the following week, in the envelope hanging outside my office (449 Birge).

In any homework exercise marked "DTB" (meaning, "Done This Before"), you may get full credit by simply stating, "DTB". Please also say where you have done it before, such as, "Math 799 at Appalachian State Teacher's College" or "Self Study".

Reading Assignment: Nakahara, Chapter 1 (for overview of applications) and pp. 67–77.

1. (DTB) A common way to obtain equivalence classes is through a group action. Let G be a group and M a space. A *group action* is an association between elements $g \in G$ and bijections $\Phi_g : M \to M$, such that $\Phi_g \Phi_h = \Phi_{gh}$. Properly speaking, the action itself is the mapping : $g \mapsto \Phi_g$.

Note that the space of bijections of M onto itself is itself a group, with composition being the multiplication law. Thus, a group action can be regarded as a group homomorphism between G and this space of bijections. In many physical applications, M is a differentiable manifold and the maps Φ_g are diffeomorphisms (terminology to be explained later). The maps Φ_g are then transformations of M onto itself (rotations, Lorentz transformations, canonical tranformations, etc).

(a) Given a group action on a space M, we may consider two points $x, y \in M$ as equivalent if $y = \Phi_g x$ for some $g \in G$. Show that this is an equivalence relation. The equivalence class $[x]$ is called the *orbit* of x under the group action; for example, think of the spheres (S^2) which result by applying rotations $SO(3)$ to a point of $\mathbb{R}^3$. Thus, the space M is broken up into disjoint subsets, the orbits of the group action.

(b) Given a group G and a subgroup H, the *left cosets* of H are the sets $[g] = \{gh|h \in H\}$, where g is the representative element of the coset. Similarly, the *right coset* of H containing g is the set $\{hg|h \in H\}$. Show that the left and right cosets are orbits of (two different) group actions of H on G (identify the respective group actions). (Left and right cosets are sometimes denoted gH and Hg, respectively.)

(c) In Nakahara's exercises 2.6 and 2.7, he wants you to show that the relation defined is an equivalence relation by appealing to the definition of an equivalence relation. Do these problems instead by showing that equivalent points lie on the orbit of some group action. In problem 2.6,

$G = SL(2, \mathbb{Z})$ and $M = H$, and in problem 2.7, G is any group that acts on itself ($M = G$) by conjugation.

Here is a confusing point regarding the terminology of group actions. The group action defined above is sometimes called a “left action.” In this course, all group actions will be left actions, so we'll omit the “left”. But for reference, here is the definition of a “right action.” A right action of a group G on a space M is an association between elements $g \in G$ and bijections Φ_g of M onto itself such that $\Phi_g \Phi_h = \Phi_{hg}$ for all $g, h \in G$ (the Φ products are in the reverse order from a left action). Any right action is closely associated with a left action (this is why we will only use left actions in this course). To see this, suppose we have a right action : $g \mapsto \Phi_g$. Then define a different mapping between the group and the same set of bijections Φ_g of M onto itself by $\Psi_g = \Phi_{g^{-1}}$. Then

$$\Psi_g \Psi_h = \Phi_{g^{-1}} \Phi_{h^{-1}} = \Phi_{h^{-1} g^{-1}} = \Phi_{(gh)^{-1}} = \Psi_{gh}. \tag{1}$$

Thus, the map : $g \mapsto \Psi_g$ is a left action.

2. Let $SO(3)$ be the usual group of proper rotations, with the usual action on $\mathbb{R}^3$. Let $SO(2)$ be the subgroup of rotations about the z-axis. Let $R \in SO(3)$ and write R in Euler angle form,

$$R = R_z(\alpha) R_y(\beta) R_z(\gamma). \tag{2}$$

Consider the left cosets of $SO(2)$ within $SO(3)$. If two rotations belong to the same left coset, what can you say about their Euler angles? What about right cosets?

Find the topology of the quotient space, $SO(3)/SO(2)$ (use left cosets). Hint: Don't try to do this in Euler angles, they are ugly. Instead, consider the action of an arbitrary rotation in $SO(3)$ on the z-axis, and use it to characterize the cosets.

3. (DTB) On p. 77 Nakahara talks about the orthogonal complement to the kernel of f. The objection to this, as discussed in class, is that you can't define an orthogonal complement without a metric, which you may not have (and which Nakahara has not yet introduced at that point in the text). Another problem with Nakahara's remark is that the space spanned by the vectors h_i (his notation) is not unique, because the h_i are not (in general) unique.

Let V be a vector space and U a vector subspace. Let $v_1 \sim v_2$ if $v_1 - v_2 \in U$. Geometrically, this means that v_1 and v_2 lie in a “plane” parallel to U. Show that $\sim$ is an equivalence relation. Let $V/\sim$ be denoted V/U. Show that V/U can be given the structure of a vector space (one can define addition of equivalence classes and their multiplication by scalars).

If we now let $f : V \to W$ be a linear map and identify U with $\ker f$, there is an obvious way to define a mapping

$$\hat{f} : \frac{V}{\ker f} \to \operatorname{im} f. \tag{3}$$

Do this, and show that the mapping is a vector space isomorphism. In an appropriate basis, $\hat{f}$ is represented by a square, invertible matrix, even though f originally may well have been represented by a noninvertible matrix, even a rectangular matrix. It is in this sense that all matrices have an inverse. But note that the domain of $\hat{f}$ is not a subspace of V, it is a quotient space.

If a metric is introduced into V, so that orthogonality is defined, then show how the subspace of V which is orthogonal to $\ker f$ can be identified with the quotient space $V/\ker f$.

4. Let V be a vector space and $U \subseteq V$ be a vector subspace. Let V^* be the dual space to V. Let $X^* \subseteq V^*$ be the space of dual vectors that annihilate U, that is, $\alpha \in X^*$ if $\alpha(u) = 0$ for all $u \in U$. Prove that

$$\dim U + \dim X^* = \dim V. \tag{4}$$

If now we have a mapping $f : V \to W$, show that

$$\dim \operatorname{im} f = \dim \operatorname{im} f^*, \tag{5}$$

where $f^* : W^* \to V^*$ is the pull-back.

A remark here is that if we have a subspace $U \subseteq V$, one way to specify U is to specify a set of vectors that span U (a basis in U). But a complementary way is to specify a complete set of covectors that annihilate U (a basis in X^*, in the notation above). U is then the simultaneous kernel of these covectors. This is an example of switching from a space to its dual to understand a problem, often an effective strategy.