Physics 221A
Fall 2020
Appendix C
Gaussian Integrals†

1. The Main Results

The most common Gaussian integral encountered in practice is

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \frac{\sqrt{\pi}}{\sqrt{a}}, \tag{1}$$

where a is real and $a > 0$. The integral does not exist (it diverges) if $a \le 0$.

Another common form is when the exponent is purely imaginary. In this case we have

$$\int_{-\infty}^{+\infty} dx\, e^{icx^2} = e^{is\pi/4}\,\frac{\sqrt{\pi}}{\sqrt{|c|}}, \tag{2}$$

where c is real and $c \ne 0$, and where $s = \operatorname{sgn} c = \pm 1$. The integral does not exist (it diverges) if $c = 0$. Notice that Eq. (2) is a special case of Eq. (1) if a is allowed to be complex, if we set $a = -ic$, and if $\sqrt{a} = \sqrt{-ic}$ is interpreted according to

$$\sqrt{-ic} = \begin{cases} e^{-i\pi/4}\sqrt{|c|}, & c > 0, \\ e^{i\pi/4}\sqrt{|c|}, & c < 0. \end{cases} \tag{3}$$

If we allow a in Eq. (1) to be an arbitrary complex number, then the integral converges exponentially if $\operatorname{Re} a > 0$. If $\operatorname{Re} a < 0$ then the integrand diverges exponentially and the integral is not defined. The case $\operatorname{Re} a = 0$ is the marginal case. If $\operatorname{Re} a = 0$ and $\operatorname{Im} a \ne 0$, then the integrand oscillates with a sequence of positive and negative lobes with decreasing area as x increases, so the integral converges. Finally, if $\operatorname{Re} a = 0$ and $\operatorname{Im} a = 0$ (that is, $a = 0$), then the integral does not converge. In summary, the integral converges everywhere to the right of the imaginary axis in the complex a-plane, and on the imaginary axis except at $a = 0$.

If a is a complex number such that the integral (1) converges, then the answer is still given by Eq. (1), as long as $\sqrt{a}$ is interpreted in the following sense. We write

$$a = |a|e^{i\phi}, \tag{4}$$

where $-\pi/2 \le \phi \le \pi/2$, and we define

$$\sqrt{a} = e^{i\phi/2}\,\sqrt{|a|}. \tag{5}$$

† Links to the other sets of notes can be found at:
`http://bohr.physics.berkeley.edu/classes/221/1920/221.html`.

This means that of the two square roots of a, we take the one with the positive real part.

Another common form is one with a linear term in the exponent,

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{b^2/4a}. \tag{6}$$

The existence of this integral depends on the value of a in the same way as the simpler integral (1), because for large x the quadratic term in the exponent dominates. That is, a can be any complex number to the right of the imaginary axis, or on the imaginary axis, except for $a = 0$. The convergence of the integral is independent of b, which can be any complex number. The quantity $\sqrt{a}$ on the right hand side of Eq. (6) is interpreted as in Eq. (5).

2. The Proof

This is the standard proof, see, for example, E. B. Wilson, *Advanced Calculus*.

First take the case where a is real and $a > 0$. By a change of variable $\sqrt{a}\,x = x'$ the answer can be expressed in terms of

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} = \int_{-\infty}^{\infty} dy\, e^{-y^2}, \tag{7}$$

where we have dropped the prime on x in the first integral. Squaring this we have

$$I^2 = \int dx\, dy\, e^{-(x^2+y^2)} = \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\theta\, e^{-\rho^2}, \tag{8}$$

where we have switched to polar coordinates in the second integral. This integral is easily done, yielding $I^2 = \pi$. Taking the square root and transforming back to the original variable of integration reproduces Eq. (1). The method is a special trick that is worth knowing because the result is so important.

To prove Eq. (1) in the case that a is complex (but in the allowed region described in Sec. 1) we do a change of variable,

$$t = \sqrt{a}\, x, \tag{9}$$

where $\sqrt{a}$ is defined in Eq. (5). The contour of integration in the original integral runs along the real axis in the complex x-plane, so after the change of variable the contour is $\arg t = \phi/2$, $\arg t = \phi/2 + \pi$ in the complex t-plane. However this contour can be deformed to run along the real t-axis, whereupon the integral is reduced to the form of Eq. (1) with $a = 1$. This is really an example of the method of steepest descent, an approximation method for integrals which however in this case is exact.

To prove the integral (6) we complete the square and shift origin to reproduce the form (1).

3. Multidimensional Gaussian Integrals

A common form of a multidimensional Gaussian integral is

$$\int d^n x \, \exp(-x^T \cdot A \cdot x + b^T \cdot x) = \frac{\pi^{n/2}}{\sqrt{\det A}} \, \exp\Bigl(\frac{b^T \cdot A^{-1} \cdot b}{4}\Bigr), \tag{10}$$

where x is a a real n-vector and the range of integration is all of $\mathbb{R}^n$, where A is a real, $n \times n$ symmetric, positive definite matrix, where b is an n-vector, where T means transpose, where x and b are seen as column vectors and x^T and b^T as row vectors, and where the dot indicates contractions or matrix multiplication. The vector b is allowed to be complex.

The case where the matrix A in Eq. (10) is purely imaginary is also common. In this case we have

$$\int d^n x \, \exp(i x^T \cdot C \cdot x + b^T \cdot x) = e^{i\nu\pi/4} \frac{\pi^{n/2}}{\sqrt{|\det C|}} \, \exp\Bigl(i \frac{b^T \cdot C^{-1} \cdot b}{4}\Bigr). \tag{11}$$

Here C is a real, $n \times n$ symmetric matrix with $\det C \neq 0$; ν is an integer,

$$\nu = \nu_+ - \nu_-, \tag{12}$$

where $\nu_\pm$ is the number of positive or negative eigenvalues of C; and otherwise the notation is as in Eq. (10).

The case where A in Eq. (10) is a complex symmetric matrix is more complicated but encountered less frequently, so we shall not consider it.

Equation (10) is proved by performing an orthogonal transformation,

$$x = R \cdot y, \tag{13}$$

where y is another real n-vector and R is an orthogonal matrix such that

$$R^T A R = \Lambda, \tag{14}$$

where Λ is diagonal. Then $d^n x = d^n y$, and the new integral breaks up into the product of one-dimensional integrals of the form (1). Equation (11) is proved similarly, except the resulting one-dimensional integrals have the form (2).

4. Moment Integrals

We present some one-dimensional integrals giving moments of Gaussian. The integral for the first moment is

$$\int dx \, x e^{-ax^2 + bx} = \frac{b}{2a} \frac{\sqrt{\pi}}{\sqrt{a}} \, e^{b^2/4a}, \tag{15}$$

while the second moment is

$$\int dx \, x^2 e^{-ax^2 + bx} = \Bigl(\frac{1}{2a} + \frac{b^2}{4a^2}\Bigr) \frac{\sqrt{\pi}}{\sqrt{a}} \, e^{b^2/4a}. \tag{16}$$

These may be obtained by differentiating Eq. (6) with respect to a or b.