\[ H = \sum_{\text{ps}} E \left( N_{\text{ps}}^{(+)} \cdot N_{\text{ps}}^{(-)} \right). \]

where

\[ N_{\text{ps}}^{(+)} = b_{\text{ps}}^+ b_{\text{ps}} \quad (+ \text{ energy}) \]
\[ N_{\text{ps}}^{(-)} = c_{\text{ps}}^+ c_{\text{ps}} \quad (- \text{ energy}) \]

We must do some work to interpret and understand the new field theory. Let's begin by computing the Heisenberg equations of motion for the b's and c's. This gives

\[ \dot{b}_{\text{ps}} = -i \left[ b_{\text{ps}}, H \right] \]

\[ = -i \sum_{p's'} E' \left( \left[ b_{p's'}, b_{p's'}^+ b_{p's'} \right] + \text{c-terms} \right) \rightarrow \text{vanishes.} \]

\[ = b_{p's'} b_{p's'}^+ b_{p's'} - b_{p's'}^+ b_{p's'} b_{p's} \text{ anticommut.} \]

\[ = - b_{p's'}^+ b_{p's'} b_{p's} - b_{p's'} b_{p's'}^+ b_{p's} + 8 p_p s_s' b_{p's'} \]

\[ = + b_{p's'}^+ b_{p's'} b_{p's} - b_{p's'} b_{p's'}^+ b_{p's} + 8 p_p s_s' b_{p's'} \]

\[ \Rightarrow \quad \dot{b}_{p's} = -i E b_{p's}, \quad \text{or} \quad b_{p's}(t) = b_{p's}(0) e^{-i E t}. \]

\[ \Rightarrow \text{sim. we find} \quad c_{p's}(t) = c_{p's}(0) e^{+i E t}. \text{ These are the} \]
same eqns and solutions we found in the classical
theory (now reinterpreted as operators). Thus the quantum
field $\Psi (x, t)$ evolves in time by the same formula above quoted
in the 1st quantized theory. (And the Dirac eqn. becomes the
Heisenberg eqn. of motion for the quantum field $\Psi$.)

The eigenstates of the Hamiltonian are specified by a string
of occupation numbers, one for each mode, \( ..., n_{ps}^\pm \) or \( \bar{n}_{ps} \),
where each $n_{ps}^\pm = 0$ or 1, and

\[
| ... n_{ps}^\pm ... \rangle = \prod_{ps} (b^\dagger_{ps})^{n_{ps}^+} \prod_{ps} (c^\dagger_{ps})^{\bar{n}_{ps}} |0\rangle
\]

where the \( \pm \) on $n_{ps}^\pm$ means # of pos. or neg. energy elections.

By having $H$ act on these states, we get

\[
H | ... n_{ps}^+ ... \bar{n}_{ps} ... \rangle = \sum_{ps} E (n_{ps}^+ - \bar{n}_{ps}) | ... n_{ps}^+ ... \bar{n}_{ps} ... \rangle.
\]

For simplicity just look at a single electron state, say $b^\dagger_{ps} |0\rangle$. Then

\[
H (b^\dagger_{ps} |0\rangle) = E (b^\dagger_{ps} |0\rangle) \quad \text{where} \ E = E(p).
\]

Similarly,

\[
H (c^\dagger_{ps} |0\rangle) = -E (c^\dagger_{ps} |0\rangle).
\]

The excitations have energy $E = E(p)$ (or $-E$).

What is their momentum? For this we need a field momentum
operator. For this we go back to the 1st quantized theory and derive a conserved momentum vector $\vec{P}$ for the Dirac field. This follows by applying Noether's theorem to the field Lagrangean $L$, which is invariant under translations. We find

$$\vec{P} = \int d^3x \; \psi^+ (-i \nabla) \psi.$$  

This is simple; it is just the expectation value (in the 1st qu. theory) of the momentum operator $-i \nabla$. Now we quantize (i.e. 2nd quantize) $\psi$ to get the field operator $\psi$. We must normalize order. We also express in terms of $b'$s and $c'$s. This gives

$$\vec{P} = \int d^3x \; : \psi^+ (-i \nabla) \psi : = \sum_{ps} \vec{P} (b_{ps}^+ b_{ps} - c_{ps}^+ c_{ps}).$$

Thus

$$\vec{P} (b_{ps}^+ |0\rangle) = \vec{P} (b_{ps}^+ |0\rangle)$$

$$\vec{P} (c_{ps}^+ |0\rangle) = -\vec{P} (c_{ps}^+ |0\rangle).$$

The excitations $b_{ps}^+ |0\rangle$ have energy $E$ and momentum $\vec{P}$, while $c_{ps}^+ |0\rangle$ has energy $+\text{mom.} -E$ and $-\vec{P}$. With this we are satisfied that the excitations should be identified with electron states. $1^{st}$ qu.

We have worked with two bilinear quantities of the classical field, $\hat{H}$ and $\hat{P}$, which we carried over to field operators. Thee
is another bilinear quantity important in the 1st quantized theory, namely the total probability:

\[
1 = \int d^3x \; \psi^\dagger \psi
\]

When we 2nd quantize \( \psi \) this becomes a field operator,

\[
\int d^3x \; : \psi^\dagger \psi : = \sum_{ps} (b_{ps}^+ b_{ps} + c_{ps}^+ c_{ps}).
\]

It is the sum of + and - energy number operators, so it represents the total # of electrons in the system (either pos or neg. energy).

If we multiply by \( \mathcal{Q} = -e \) we get a charge operator,

\[
\mathcal{Q} = -e \int d^3x \; : \psi^\dagger \psi : = -e \sum_{ps} (b_{ps}^+ b_{ps} + c_{ps}^+ c_{ps})
\]

At this point we have a 2nd quantized version of Dirac's theory of pos and neg. energy elections. It gives the correct FD statistics in multiparticle problems, but otherwise its physical content does not go beyond what we had with the 1st quantized theory. In particular, it does not address the interpretational difficulties of the neg. energy solutions.

To fix this up, we borrow ideas from hole theory. If all the negative energy states are filled, as Dirac supposed, then when we
excite an electron out of a negative energy state, we create a hole. Thus, the destruction of a neg. energy electron in the sea is equivalent to the creation of a hole (or position). Thus let us define

\[ C_{\psi} = d_{\psi}^{+} \]

where \( d_{\psi}^{+} \) creates a position of charge, energy, momentum and spin \( +e, +E, +\vec{p} \) and \( +\mathbf{s} \). Likewise, if an electron makes a radiative transition to an unoccupied neg. energy state (a hole), it creates a neg. energy electron or destroys a hole. So let us write

\[ C_{\psi}^{+} = d_{\psi}. \]

Note that with these def's, the \( d_{\psi}, d_{\psi}^{+} \) satisfy anticommutation relations exactly like the \( b's \) and \( c's \),

\[ \{d_{\psi}, d_{\psi}^{+}\} = \delta_{pp'}\delta_{ss'} \]

all other \( \delta_{\psi}, \tilde{\delta} = 0. \)

Now the quantum field is

\[ \psi(x') = \frac{1}{\sqrt{V}} \sum_{ps} \frac{1}{\sqrt{m/E}} \left( b_{ps} u(ps) e^{i\vec{p}\cdot\vec{x}'} + d_{ps}^{+} v(ps) e^{-i\vec{p}\cdot\vec{x}'} \right). \]

Also, the field Hamiltonian becomes,
\[ H = \sum_{ps} E (b^+_p b^-_{ps} - d^+_p d^-_{ps}). \]

Notice the ordering of the \( d \) operators. If we anticommutate them, we get

\[ H = \sum_{ps} E (b^+_p b^-_{ps} + d^+_p d^-_{ps}) - \sum_{ps} E \]

The final term is \( \infty \). One interpretation is that it is the \( \infty \) (neg.) energy of the filled Dirac sea. Another interp. is that it is an \( \infty \) term resulting from the ordering ambiguities on passing from the classical expression to a qu. operator. That is, it is a zero point term.

We throw away zero point terms to make vacuum expectation values vanish. But is the vacuum the state without any electrons of any kind, pos or neg. energy, or is it the state without either electrons or positions? The latter is more physical, and if we want \( \langle 0 | H | 0 \rangle = 0 \) for this vacuum, then we must throw away the term \( -\sum_{ps} E \) above.

This leads to a new interpretation of normal ordering: We migrate \( b^+ \)'s and \( d^+ \)'s to the left, \( b^- \)'s and \( d^- \)'s to the right, keeping any sign changes on applying anticommutators, but
throwing away the anti-commutators themselves. This differs from what we did above in that we use $d^+, d^{-}$ not $c^+, c^{-}$.

Thus,

$$\mathcal{H} = \int d^4x : \psi^+ (\mathbf{- i \partial} \cdot \mathbf{\nabla} + m \beta) \psi : = \sum_{ps} E (b_{ps}^+ b_{ps} + d_{ps}^+ d_{ps})$$

$$\mathbf{P} = \int d^4x : \psi^+ (-i \mathbf{\nabla}) \psi : = \sum_{ps} \mathbf{P} (b_{ps}^+ b_{ps} + d_{ps}^+ d_{ps})$$

$$Q = \int d^4x : \psi^+ \psi : = -e \sum_{ps} (b_{ps}^+ b_{ps} - d_{ps}^+ d_{ps})$$

Notice the sign changes. Now positions have positive energy, momentum (and spin), but the charge operator is no longer positive definite. The quantity Dirac worked so hard to make positive definite in the 1st quantized theory is now replaced by an operator of either sign in the 2nd quantized theory. Energies are strictly positive, and we deal strictly with observable objects (electrons and positions). Dirac's original goal of "curing" the "problems" with the KG eqn now appears as irrelevant, although there was no way to see that within the framework of the single particle (1st quantized) theory. The real conceptual breakthrough came with hole theory.
Now consider the Dirac electron field (and quantized) interacting with the EM field. Initially for simplicity we take the EM field as a specified $\phi$-number field $A_\mu(x,t)$. So we don't need to worry about photons or Maxwell equs.

To obtain the form of interaction, we go back to the free Dirac Lagrangian and use the minimal coupling prescription,

$$L = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi = L_{\text{free}} \quad \text{(free particle)}$$

$$i \gamma^\mu \rightarrow i \sigma_\mu \pi \gamma^\mu,$$

$$L = L_{\text{free}} + L_{\text{int}},$$

$$L_{\text{int}} = - g \bar{\Psi} \gamma^\mu A_\mu \Psi = - g \bar{\Psi} \gamma^\mu A_\mu \Psi = - A_\mu J^\mu$$

where $J^\mu = g \bar{\Psi} \gamma^\mu \Psi$ is the current. Then for the Hamiltonian density we get

$${\mathcal{H}} = \pi \Psi + \bar{\pi} \bar{\Psi} - L$$

$$= H_{\text{free}} + H_{\text{int}},$$

$$H_{\text{free}} = \Psi^\dagger (-i \gamma^\mu \partial_\mu + m\beta) \Psi$$

$$H_{\text{int}} = - L_{\text{int}} = g \bar{\Psi} \gamma^\mu A_\mu \Psi.$$