Some topics covered in lecture but not in the typewritten notes.

The Lippmann-Schwinger equa,

$$|\psi_2\rangle = |\phi_2\rangle + G_0^+(E)V|\phi_2\rangle,$$

is not a solution for the scattering state $|\psi_2\rangle$, because this state appears on both the LHS and RHS. (The state $|\phi_2\rangle$ is the free particle state with wave function

$$\langle \phi_2 | \phi_2 \rangle = \frac{e^{-i \mathbf{p} \cdot \mathbf{r}}}{(2\pi)^{3/2}}$$

and normalization

$$\langle \phi_2 | \phi_2 \rangle = 8^{3/2} (2\pi)^{3/2}.\)$$

Rewrite the LS equa,

$$|\phi_2\rangle = [1 - G_0^+(E)V]|\psi_2\rangle,$$

and it's clear we need to invert $1 - G_0^+(E)V$ to solve for $|\phi_2\rangle$. If $V$ is "small", we can hope that a power series expansion will work,

$$[1 - G_0^+(E)V]^{-1} = 1 + G_0^+(E)V + G_0^+(E)V G_0^+(E)V + \ldots.$$ 

This gives

$$|\psi_2\rangle = |\phi_2\rangle + G_0^+(E)V|\phi_2\rangle + G_0^+(E)V G_0^+(E)V|\phi_2\rangle + \ldots$$

(this is called the Born series. If it is a good approximation, then the scattered wave is small compared to the incident
wave, and \(|\Psi_0\rangle \approx |\Psi_k\rangle\). Usually it is most important that this condition hold where \(V(k)\) is largest; this is also where the scattered wave is largest.

Thus we find the intuitive conditions of validity of the Born approximation: if \(V\) is small enough that the exact wave is nearly the same as the incident wave where \(V(k)\) is large, or if the incident energy is high enough that the incident wave is not distorted much by the potential. The Born approximation works best at high energy.

The Born series truncated at the first 2 terms is the first Born approximation for the wave field; the higher order terms give the 2nd, 3rd etc. Born approximation for the wave field.

The full Born series can be used in the exact expression for the scattering amplitude,

\[ f(k, k') = -(2\pi)^2 \frac{m}{\hbar^2} \langle \phi_k | V | \phi_k \rangle, \]

to give

\[ f(k, k') = -(2\pi)^2 \frac{m}{\hbar^2} \left[ \langle \phi_k | V | \phi_k \rangle + \langle \phi_k | V G_0(k) V | \phi_k \rangle + \cdots \right] \]

The 1st term is the first Born approx for \(f\), etc.
The matrix element in the 1st Born appx is just the Fourier transform of the potential, evaluated at the momentum transfer $\mathbf{q} = \mathbf{q}' - \mathbf{q}$:

$$\langle \phi_{\mathbf{q}'} | V | \phi_{\mathbf{q}} \rangle = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} e^{-i (\mathbf{q}' - \mathbf{q}) \cdot \mathbf{p}} \tilde{V} (\mathbf{p})$$

$$= \frac{1}{(2\pi)^3} \tilde{V} (\mathbf{q}' - \mathbf{q})$$

using our conventions for Fourier transforms. Thus the cross section is

$$\frac{d\sigma}{d\Omega} = | f(\theta, \phi) |^2 = 2\pi \frac{m_e^2}{\hbar^4} | \tilde{V} (\mathbf{q}' - \mathbf{q}) |^2.$$ 

This is the same answer as from time-dep. perturbation theory. At higher Born approximations, $d\sigma/d\Omega$ no longer depends only on $\mathbf{q}' - \mathbf{q}$.

As an example of the Born approximation, we consider the Yukawa potential,

$$V(r) = \frac{A}{r} e^{-\frac{r}{\lambda}},$$

where $\lambda$ = the range of the potential and $A$ = a const. This potential describes the strong interactions by means of pion exchanges, a simple model by today's standards but still useful at low energies. The Yukawa potential is the Greens fn. for the static
\( \frac{a}{\alpha} = 0 \) interaction mediated by a particle of mass \( m \), where

\[
a = \frac{\hbar}{mc} \quad \text{Compton wavelength of particle.}
\]

If \( a \to \infty \) \( (m \to 0) \) we get the Coulomb potential, for massless photons.

To get the 1st Born approx. to the cross section, we need the Fourier transform,

\[
\tilde{V}(q^2) = \frac{4\pi A}{(2\pi)^{3/2}} \frac{a^2}{1 + a^2 q^2},
\]

so

\[
\frac{d\sigma}{d\Omega} = \frac{4A^2 m^2}{k^4} \frac{a^4}{(1 + a^2 q^2)^2}
\]

Note, \( q^2 = (\vec{k}' - \vec{k})^2 = 2k^2(1 - \cos\theta) = 4k^2 \sin^2 \frac{\theta}{2} \),

\[
\vec{k}' \xrightarrow{\theta} \vec{k}, \quad \text{where } k = k' \text{ by conservation of energy.}
\]

So,

\[
\frac{d\sigma}{d\Omega} = \frac{4A^2 m^2}{k^4} \left[ \frac{a^4}{1 + 4k^2 a^2 \sin^2 \frac{\theta}{2}} \right]^2.
\]

This depends explicitly on \( kA \). Recall that \( kA \) is the approximate cutoff value of \( l \) in the partial wave series. Look at limits of this cross section.
Consider case $ka \ll 1$. Then the 1 dominates in the denominator, and

$$\frac{\Delta \sigma}{\Delta \Omega} = \frac{4A^2 \mu^2 a^4}{h^4}.$$

The cross section is isotropic (s-wave scattering), as we expect when $ka \ll 1$. Of course this doesn’t mean the cross section has the right value, merely that it has the right angular dependence.

In the case $ka \gg 1$, the result depends on the angle. If $\theta$ is not too small, then $k^2 a^2 \sin^2 \theta/2 \gg 1$ and this term dominates the denominator. Then the cross section becomes the Rutherford cross section, and is independent of $a$ (goes like $1/\sin^4 \theta/2$). But at angles such that $\theta \approx 1/ka$ (small angles), the 1 in the denominator takes over, and prevents $d\sigma/d\Omega$ from diverging at small $\theta$. The forward peak (small $\theta$) has a width $\sim 1/ka$, which again agrees with the idea that the partial wave series is cut off after $2\approx ka$ terms (hence it can resolve angles down to $\Delta \theta \approx 1/ka$).

Sketch:

\[ \Delta \theta \sim \frac{1}{ka} \quad \text{and} \quad \sim \frac{1}{\sin^4 \theta/2} \]
Now a more quantitative condition of validity of the Born approximation. Since we require $\Psi_\text{R}(\vec{r}) \approx \phi_\text{R}(\vec{r})$, define

$$C = \frac{\phi_\text{R}(\vec{r}) - \Psi_\text{R}(\vec{r})}{\phi_\text{R}(\vec{r})},$$ (dim-less)

which we want to be small. This depends on $\vec{r}$, but let's evaluate at $\vec{r} = 0$ for a typical (hopefully) point inside the scatterer. Then $C$ depends only on $\vec{k}$. Using the Lipmann-Schwinger eqn., we get (evaluating kernel of integral transform at $\vec{r} = 0$):

$$C(\vec{k}) = \frac{1}{i2\pi} \frac{m}{\hbar^2} \int d^3\vec{r}' \frac{e^{i\vec{k}\cdot\vec{r}'}}{r'} V(\vec{r}') \psi_\text{R}(\vec{r}') \times (2\pi)^{3/2}.$$

Since this is only an estimate, replace $\psi_\text{R}(\vec{r}')$ by $\phi_\text{R}(\vec{r}')$,

$$C(\vec{k}) = \frac{1}{i2\pi} \frac{m}{\hbar^2} \int d^3\vec{r} \frac{e^{i\vec{k}\cdot\vec{r}}}{r} V(\vec{r}) e^{i\vec{k}\cdot\vec{r}},$$

dropping primes on the dummy variable of integration. If $V(\vec{r}) = V(r)$ is central force, then we can do the angular integration, obtaining

$$C(\vec{k}) = \frac{2m}{\hbar^2} \frac{1}{k} \int_0^\infty dr e^{ikr} \sin kr V(r).$$ (<<1 demand)

This integral is small if $V(r)$ is small, or if $k$ is large, confirming our intuition about the validity of the Born approx.

Note when $k$ gets large, $C(k) \to 0$ both because of $1/k$ and because
The integrand oscillates more and more rapidly as $k \to \infty$.

If $ka \ll 1$, then the integral above gives for Yukawa,

$$C(k) = \frac{2m}{k^2} \frac{1}{4} \int_0^\infty dr \, kr \, Ae^{-r/k} = \frac{2mAk}{k^2} \ll 1.$$  

This is the condition that the Yukawa potential should be so weak that it does not support any bound states. This is the usual condition at low energy for the validity of the Born approx.

Note: Conditions of validity of 1st Born are never met (any $k$) for $a \to \infty$, the Coulomb limit. It is an accident that you get the Rutherford $\alpha/\beta$ in this case.

Now we turn to the optical theorem. First, an almost 1-line proof for central force potentials, using method of phase shifts.

We have:

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{ikr} \sin \delta_l \, P_l(\cos \theta)$$

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l.$$  

Evaluate $f(\theta)$ in the forward direction ($\theta = 0$), where $P_l(1) = 1$:

$$f(\theta = 0) = \frac{1}{k} \sum_l (2l+1) \left[ \sin \delta_l \cos \delta_l + i \sin^2 \delta_l \right],$$

so

$$\frac{4\pi}{k^2} \sin f(\theta = 0) = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l = 0.$$
In summary,
\[ \sigma = \frac{4\pi}{k} \ln f(0) \] (Optical theorem).

This is easy, but it doesn't give any insight into the meaning.
(The total cross section is proportional to the sum part of the
scattering amplitude in the forward direction.)

Now a longer proof that carries more insight. Also
generalizes to non-central force potentials. We consider the
probability current,
\[ \mathbf{j} = \frac{-i\hbar}{2m} \nabla \psi \psi^* + \text{c.c.} \]

in QM (scalar particle). For an scattering problem, \( H \mid \psi \rangle = E \mid \psi \rangle \)
it satisfies \( \nabla \cdot \mathbf{j} = 0 \), hence its integral over any closed surface
is 0. We will integrate it over a large sphere of radius \( r \)
centered on the scatterer, and take \( r \to \infty \).

So,
\[ \sigma = \int \mathbf{j} \cdot d\mathbf{a} = r^2 \int J_r \, d\Omega \]

\[ J_r = \mathbf{r} \cdot \mathbf{j}. \]
since we are in the asymptotic regime, we can take the asymptotic form of $\Psi$,
nonalized const.

$$\Psi(\vec{r}) = A \left[ e^{i\vec{k} \cdot \vec{r}} + f(\theta, \phi) \frac{e^{ikr}}{r} \right] + \text{higher orders}.$$ 

With $\vec{k} = k \hat{z}$, we have $\vec{k} \cdot \vec{r} = kr \cos \theta$. Then

$$\vec{F} = \frac{iA^2 v}{2} \left[ e^{-ikr \cos \theta} + f^* e^{-ikr} \right] \left[ e^{i \vec{k} \cdot \hat{z}} \hat{z} + f e^{ikr} \hat{r} \right] + \text{c.c.,}$$

where we drop terms that $\to 0$ faster than $1/r^2$, since they won't contribute to the integral. $\vec{F}$ breaks up into the incident, scattered, and cross-term (x) or interference currents, $\vec{F}_{\text{inc}}, \vec{F}_{\text{sc}}, \vec{F}_{\text{x}}$. Here

$$\frac{\vec{F}}{\text{inc}} = \frac{iA^2 v}{2} \hat{z} = \text{const.,}$$

so

$$\int_{\text{sphere}} \vec{F}_{\text{inc}} \cdot d\vec{a} = 0.$$

The incident flux goes right through the sphere. As for $\vec{F}_{\text{sc}},$

$$\vec{F}_{\text{sc}} = iA^2 v \frac{|f|^2}{r^2} \hat{r}, \quad |f|^2 = \frac{d\sigma}{d\Omega}$$

so

$$\int_{\text{sphere}} \vec{F}_{\text{sc}} \cdot d\vec{a} = iA^2 v \int d\Omega \frac{d\sigma}{d\Omega} = iA^2 v \sigma.$$

This is the positive flux of scattered particles.

Of course it's proportional to $\sigma$; this is the def. of $\sigma$. 
The scattered particles are coming out of the sphere, so there must be some net flux in. It must come from the cross terms.

\[ \vec{J}_x = \frac{|A|^2 v}{2} \left\{ e^{i \vec{k} \cdot \hat{r} (1 - \cos \theta)} \frac{d}{d \hat{r}} \hat{r} + e^{-i \vec{k} \cdot \hat{r} (1 - \cos \theta)} \frac{d}{d \hat{r}} \hat{r} \right\} + c.c. \]

\[ = \frac{|A|^2 v}{2} \left\{ e^{i \vec{k} \cdot \hat{r} (1 - \cos \theta)} \frac{d}{d \hat{r}} (\hat{r} \cdot \hat{r}) \right\} + c.c. \]

So,

\[ \int_{\text{sphere}} \frac{\vec{J}_x \cdot d\hat{r}}{r} = \frac{|A|^2 v}{2} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \left\{ e^{i \vec{k} \cdot \hat{r} (1 - \cos \theta)} f(\theta, \phi) (1 + \cos \theta) \right\} + c.c. \]

where we use \((\hat{r} + \hat{z}) \cdot \hat{r} = 1 + \cos \theta\). We want this in the limit \(r \to 0\), which is tricky since the integral is multiplied by \(r\) but the integrand oscillates even more rapidly as \(r \to 0\). It helps to clarify this if we set integrate by parts, using

\[ \sin \theta d\theta e^{i \vec{k} \cdot \hat{r} (1 - \cos \theta)} = \frac{1}{i \vec{k} r} \frac{d}{d \theta} \left\{ e^{i \vec{k} \cdot \hat{r} (1 - \cos \theta)} \right\} \]

So,

\[ \int_{\text{sphere}} \frac{\vec{J}_x \cdot d\hat{r}}{r} = \frac{|A|^2 v}{2} \left\{ e^{i \vec{k} \cdot \hat{r} (1 - \cos \theta)} f(\theta, \phi) (1 + \cos \theta) \right\} \bigg|_0^{2\pi} \]

\[ - \int_0^\pi d\theta \left\{ \frac{d}{d \theta} \left[ f(\theta, \phi) (1 + \cos \theta) \right] \right\} + c.c. \]
The r's out front cancel and the remaining integral still has the oscillating integrand, so it goes to 0 as \( r \to \infty \), and we just have the 1st term,
\[
e^{-ikr(1-\cos \theta)} f(\theta, \phi)(1+\cos \theta) \bigg|_0^\pi = 0 - 2f(0, \phi).
\]

But \( f(0, \phi) \) is \( f \) at the north pole, where it doesn't depend on \( \phi \). Just call it \( f(0) \). So the \( \phi \) integral can be done, and gives \( 2\pi \).

Thus,
\[
\int \vec{F}_x \cdot d\vec{a} = \frac{1A^2 v}{2} \frac{2\pi}{k} \left[ -\frac{2f(0)}{i} + \frac{2f(0)^*}{i} \right]
\]
\[
= 1A^2 v \frac{4\pi}{k} \text{Im } f(0).
\]

Putting it together, \( \int \vec{F} \cdot d\vec{a} = 0 \) \( \Rightarrow \)

\[
\sigma = \frac{4\pi}{k} \text{Im } f(0)
\]

We see that the scattered flux is compensated by an inward coming flux from down the + \( z \)-axis, coming from the interference between the incident and scattered wave. This flux cancels out some of the incident flux, creating the shadow shadow downstream from the scatterer. More later...