This problem is a variation on Sakurai, problem 7.1.

(a) Find the free-particle Green’s function in one dimension,

\[ G_0^+ (x, x'; E + i\epsilon) = \langle x | \frac{1}{E + i\epsilon - H_0} | x' \rangle, \tag{1} \]

where

\[ H_0 = \frac{p^2}{2m}. \tag{2} \]

Also find \( G_0^- (x, x'; E - i\epsilon) \). Show explicitly that they satisfy

\[ \lim_{\epsilon \to 0} \left( E + \frac{h^2}{2m} \frac{d^2}{dx^2} \right) G_0^+ (x, x'; E \pm i\epsilon) = \delta(x - x'). \tag{3} \]

Show that \( G_0^- \) is the analytic continuation of \( G_0^+ \) across the negative real energy axis, and that there is a branch cut across the positive real energy axis. Analytically continue \( G_0^+ \) or \( G_0^- \) onto the second Riemann sheet. Does the Green’s function have any singularities (on either sheet) apart from the branch point at \( E = 0 \)?

(b) Write down a 1-dimensional version of the Lippmann-Schwinger equation for an exact scattering solution \( \psi(x) \) associated with an incoming (from the left) free particle state \( \phi(x) = e^{ikx}/\sqrt{2\pi} \). The exact solution \( \psi(x) \) satisfies the Schrödinger equation in a potential \( V(x) \), which you can consider to be localized. Consider asymptotic forms (large \( |x| \)) and find expressions for the transmission and reflection amplitudes \( t \) and \( r \) which are analogous to Eq. (28.68) in three dimensions. These amplitudes are defined by

\[ \psi(x) = \frac{1}{\sqrt{2\pi}} [e^{ikx} + re^{-ikx}], \quad (x \to -\infty), \]

\[ \psi(x) = \frac{1}{\sqrt{2\pi}} te^{ikx}, \quad (x \to +\infty). \tag{4} \]
(c) Consider the potential,

\[ V(x) = \lambda \delta(x). \]  

This potential can be seen as the limit of a rectangular barrier (for \( \lambda > 0 \)) with width \( a \) and height \( V_0 = \lambda/a \) as \( a \to 0 \). The attractive case \( \lambda < 0 \) is similar.

Solve the Lippmann-Schwinger equation directly, and write out explicit forms for the wave function \( \psi(x) \) for \( x < 0 \) and \( x > 0 \). To help the reader, please use the abbreviation,

\[ D = \frac{m \lambda}{\hbar^2 k}, \]  

as much as possible. Note that \( D \) is dimensionless. Compute \( t \) and \( r \) in terms of \( D \) and show explicitly that \( |t|^2 + |r|^2 = 1 \).

(d) The operator equation,

\[ G^+ = G_0^+ + G_0^+ V G^+, \]  

is easily proved. It is a kind of Lippmann-Schwinger equation for the exact Green’s function. Write this out as an integral equation for the exact Green’s function, assume the potential is given by (5), and solve for \( G_+(x, x'; E + i\epsilon) \). Consider the case \( \lambda < 0 \). Show that this Green’s function has one pole on the negative energy axis, located at the energy of the one (and only) bound state. Show that the residue of this pole is the projection operator onto the eigenspace of this bound state.

2. Sakurai, problem 7.2.

3. Consider \( N \) static spherically symmetric scattering centers placed on a straight line such that the \( n \)-th scatterer is at point \((n-1)a\), for \( n = 1, \ldots, N \). A particle with incident momentum \( \hbar k \) such that \( k \cdot a = 0 \) is scattered from the array. Assuming the validity of the Born approximation, show that the elastic differential cross section is of the form,

\[ \frac{d\sigma}{d\Omega} = |F(\alpha)|^2 \frac{d\sigma_0}{d\Omega}, \]

where \( d\sigma_0/d\Omega \) is the differential cross section for scattering by a single scatterer. Find the form factor \( F(\alpha) \), and the geometrical meaning of the angle \( \alpha \).