Physics 221B
Spring 2007
Homework 5
Due Thursday, February 22, 2007

Reading Assignment: Lecture notes for the week, plus Reprint 1, which is from Ballentine, *Quantum Mechanics: A Modern Development* (World Scientific, 1998). This reprint contains a discussion of the definition of a cross section, how to transform the cross section from the center of mass frame to the lab frame and other topics not covered in class, as well as the method of partial waves and phase shifts for central force scattering. See also Sakurai pp. 399-416 on central force scattering, including a discussion of hard sphere scattering and scattering lengths.

1. Short-range potentials give rise to \( s \)-wave scattering at sufficiently low energies, as discussed in class. The condition is \( \lambda \gg R \), where \( \lambda \) is the de Broglie wavelength of the incident wave, and \( R \) is the range of the potential. In this case only the one term \( \ell = 0 \) contributes to the partial wave expansion of the scattering amplitude, and the amplitude itself is characterized by a single parameter, the \( \ell = 0 \) phase shift \( \delta_0 \). Any other short range potential with the same value of \( \delta_0 \) will behave the same insofar as low energy scattering is concerned. For this reason we often replace a real potential by a delta function, multiplied by some strength parameter \( g \) that we can adjust to make the phase shift \( \delta_0 \) come out right, since this is mathematically simpler than the true potential. This partly explains the popularity of \( \delta \)-function potentials in theoretical models.

In Bose-Einstein condensates (cold gases of bosonic atoms), the temperature and density are such that for atom-atom scattering \( \lambda \gg R \), where \( R \) is the radius of the atom and where \( \lambda \), the de Broglie wavelength, is comparable to the interparticle separation (this is required for the condensation). Thus replacing the atom-atom potential by a \( \delta \)-function is a good approximation.

Typically Bose-Einstein condensates are gases in which the atoms are moving in some external, confining potential, call it \( V(\mathbf{r}) \). Based on what we have said, the Hamiltonian for a system of \( N \) identical bosonic atoms of spin 0 in the potential \( V \) is

\[
H = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + V(\mathbf{r}_i) \right) + g \sum_{i<j} \delta^3(\mathbf{r}_i - \mathbf{r}_j). \tag{1}
\]

This differs from the Hamiltonian we used in studying the electrons in atoms in three
respects: (1) the external potential is \( V \) instead of \( Z/r \); (2) the interaction potentials are \( \delta \)-functions instead of Coulomb interactions; (3) the particles are spin-0 bosons.

Write down a trial wave function for the ground state of this system of bosons that is as close in spirit as possible to the Hartree-Fock trial wave function in atoms, given that these are bosons instead of fermions. Derive a self-consistent, three-dimensional Schrödinger-like equation that must be satisfied to minimize the energy of the system. This equation is the starting point of much current research into Bose-Einstein condensates.

2. WKB theory is an effective way of studying the asymptotic behavior of the solutions of the radial wave equation, which is important in scattering by central force potentials. As you know, the radial wave equation looks like a 1-dimensional Schrödinger equation,

\[
-\frac{\hbar^2}{2m} \frac{d^2 f}{dr^2} + U(r)f(r) = Ef(r),
\]

except that \( r \) ranges from 0 to \( \infty \), and the potential \( U(r) \) is the sum of the centrifugal potential and the true potential \( V(r) \),

\[
U(r) = \frac{\ell(\ell + 1)\hbar^2}{2mr^2} + V(r).
\]

The radial function \( R(r) \) that appears in the separation of variables, \( \psi(r) = R(r)Y_{\ell m}(\Omega) \), is related to \( f(r) \) by \( f(r) = rR(r) \).

One-dimensional WKB theory can be applied to the radial wave equation (2). It can be shown that more accurate results are obtained in the WKB treatment if the quantity \( \ell(\ell + 1) \) in the centrifugal potential is replaced by \( (\ell + \frac{1}{2})^2 \). This is called the Langer modification. Just accept this fact for the purposes of this problem; the justification has to do with the singularity of the centrifugal potential as \( r \to 0 \).

The WKB solution has the form,

\[
f(r) = \frac{1}{\sqrt{p_r(r)}} \cos \left[ \frac{S(r)}{\hbar} - \frac{\pi}{4} \right],
\]

where \( p_r(r) \) is the radial momentum,

\[
p_r(r) = \sqrt{2m[E - U(r)]},
\]

where

\[
S(r) = \int_{r_0}^{r} dp_r(r),
\]

and where \( r_0 \) is the radius of the classical turning point, that is, the root of

\[
U(r_0) = E.
\]
(a) Take the case of a free particle, $V(r) = 0$. Show that the turning point in the centrifugal potential is the same as the classical impact parameter of a particle with energy $E$ and angular momentum $(\ell + \frac{1}{2})\hbar$. For the rest of this problem, call this quantity $b$ (the usual symbol for impact parameters in classical scattering theory). Express your formulas in terms of $k$ rather than $E$, where $E = \hbar^2 k^2 / 2m$, since that simplifies them. Then show that the asymptotic (large $r$) form of the WKB solution is a sine wave with a phase shift. Reconcile this with Sakurai’s Eq. (A.5.15).

(b) If the true potential $V(r)$ falls off less rapidly than the centrifugal potential, then the true potential dominates at large $r$. Suppose that $V(r) \sim A/r^p$, where $\sim$ means, “equal to, plus terms that go to zero as $r \to \infty$ more rapidly than the terms shown,” where $A$ is a constant, and $p$ is a power which we put in the range $0 < p < 2$ so that $V(r)$ will go to zero as $r \to \infty$, but more slowly than the centrifugal potential. Show that for $p \leq 1$, the phase shift $\delta_\ell$ is not defined, that is, the asymptotic phase of the true radial wave function does not approach the free particle phase plus some constant. Show that for $p = 1$ (the Coulomb potential) the phase difference (exact-free) depends logarithmically on $r$. Show that for $1 < p < 2$, the phase shift $\delta_\ell$ is defined.

(c) If the true potential goes to zero more rapidly than the centrifugal potential, then at sufficiently large radius the centrifugal potential dominates. Let us choose $\ell$ large enough that $b$ (the turning point computed with respect to the centrifugal potential only) is far enough out that the centrifugal potential dominates there. Then the turning point in the total potential (centrifugal plus true) is approximately the same, say $r_0 = b + \delta r_0$, where $\delta r_0$ is small. Assume $V(r) \sim A/r^p$, where $p > 2$. Obtain an integral expression for the approximate phase shift $\delta_\ell$. Find the dependence of the phase shift $\delta_\ell$ on $\ell$ for large $\ell$. For this purpose you do not need to do any integrals, although they are, in fact, doable.

The coefficients of a Fourier (or Legendre) series fall off exponentially with mode number, for large mode numbers, if the function being expanded is analytic. If they fall off only as a power law, however, it means that the function has a discontinuity in one of its derivatives (the higher the power, the higher the derivative). This problem shows that for potentials with power law tails, the scattering amplitude has a discontinuity somewhere in one of its derivatives. The somewhere is not hard to guess: it is in the forward direction. Short range potentials (those that fall off exponentially or faster) have scattering amplitudes that are analytic in the scattering angle.