Physics 221B
Spring 2007
Homework 13
Due Thursday, May 3, 2007

Reading Assignment: Reprint 3 entire, Reprint 4 omitting Sec. 3.2. (We will come back to projection operators if we need them.) Also look over Sakurai, pp. 100–119, not for details but for another perspective on some of the same topics covered by Bjorken and Drell. My lecture on the Zitterbewegung was based on Sakurai, pp. 115–117.

1. It was shown in class that the angular momentum of the Dirac electron is

\[ \mathbf{J} = \mathbf{x} \times \mathbf{p} + \frac{\hbar}{2} \Sigma, \]  

(1)

where \( \Sigma \) is a 3-vector of Dirac matrices defined by

\[ \Sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma^{jk}, \]  

(2)

where \( \sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu] \). Beware: Bjorken and Drell write \( \sigma \) instead of \( \Sigma \), thereby confusing the 2 \( \times \) 2 Pauli matrices with 4 \( \times \) 4 Dirac matrices. The notation \( \Sigma \) is Sakurai’s. Consider a Dirac electron moving in a central, electrostatic potential, \( \Phi = \Phi(\mathbf{r}) \) (for example, the hydrogen atom or the free particle). Show that \( \mathbf{J} \) commutes with the Hamiltonian, but that the orbital and spin angular momentum separately do not.

2. Show that the quantity \( \bar{\psi}(x)\gamma_5 \gamma^\mu \psi(x) \) transforms as a pseudovector under Lorentz transformations.

3. Consider a Dirac electron \( (q = -e) \) in a uniform magnetic field, \( \mathbf{B} = B_0 \mathbf{\hat{z}} \). Choose the gauge,

\[ \mathbf{A} = B_0 x \mathbf{\hat{y}}, \]  

(3)

which is translationally invariant in the \( y \)-direction. This means that \( p_y \) will be a constant of the motion (although you must distinguish between the kinetic and canonical momentum).

Here is some background on the nonrelativistic problem, which will save you some time. Ignore the spin and the motion in the \( z \)-direction, for simplicity. Then the Schrödinger equation is

\[ \frac{1}{2m} \left[ \hat{p}_x^2 + \left( \hat{p}_y + m \omega x \right)^2 \right] \psi(x, y) = E \psi(x, y). \]  

(4)
Here we put hats on the momentum operators to distinguish them from the corresponding eigenvalues (where relevant), which are \( \epsilon \)-numbers. For example, \( \hat{p}_y = -i\hbar \partial / \partial y \). Also, we define

\[
\omega = \frac{eB_0}{mc}.
\]

Then the wave equation (4) is separable, and has the solution,

\[
\psi(x, y) = e^{ip_yy/\hbar} u_n(\xi),
\]

where \( p_y \) is the eigenvalue of \( \hat{p}_y \), where

\[
\xi = x + \frac{p_y}{m\omega},
\]

and where \( u_n \) is the usual, normalized Hermite function for the one-dimensional harmonic oscillator with frequency \( \omega \),

\[
 u_n(\xi) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{n!2^n}} H_n \left( \sqrt{\frac{m\omega}{\hbar}} \xi \right) \exp \left( -\frac{m\omega x^2}{2\hbar} \right).
\]

Here \( H_n \) is the usual Hermite polynomial, defined by Eq. (7.42). The energy eigenvalue for the eigenfunction (6) is

\[
E = (n + \frac{1}{2})\hbar \omega,
\]

where \( n = 0, 1, 2, \ldots \) is the Landau level. The energy is independent of the quantum number \( p_y \). The wavefunction is like a ridge in the \( x-y \) plane, centered on \( x = -p_y/m\omega \) (i.e., on \( \xi = 0 \)), and running in the \( y \)-direction.

(a) Solve the Dirac equation for the relativistic electron in the same magnetic field. This time you must include the \( z \)-motion and the spin. Express the energy in terms of the quantum numbers \( (n, p_y, p_z, m_s) \). Write out explicitly a complete set of positive energy solutions as 4-component spinors. You need not normalize these solutions, and you may ignore the negative energy solutions.

(b) Consider the motion of a Dirac electron in the field,

\[
\mathbf{B} = B_1 \hat{z}, \quad \mathbf{E} = E_1 \hat{x},
\]

where \( 0 < E_1 < B_1 \). The solution of the Dirac equation for this problem can be obtained from the solution to part (a) by using the transformation law (35.88). Find matrices \( \Lambda \) and \( D(\Lambda) \) which will cause \( \psi'(x) \) to be the solution in the field (10) if \( \psi(x) \) is the solution in the purely magnetic field of part (a). You will also need to find a relation between \( (E_1, B_1) \) and \( B_0 \). You need not write out the solution \( \psi'(x) \) explicitly, but do find the energy eigenvalues in terms of the same quantum numbers \( (n, p_y, p_z, m_s) \) as in part (a).