The Feynman path integral is an expression for the propagator in terms of an integral over an infinite-dimensional space of paths in configuration space. It constitutes a formulation of nonrelativistic quantum mechanics which is alternative to the usual Schrödinger equation. Whereas the Schrödinger equation is based on Hamiltonians, the Feynman path integral is based on Lagrangians. In the last twenty years or so, path integrals have proven to be invaluable in quantum field theory, statistical mechanics, condensed matter physics, and other areas. The path integral is now the preferred method for quantizing gauge fields, as well as setting up perturbation expansions in quantum field theory. It also leads very quickly to important conclusions in certain problems. However, for most simple nonrelativistic quantum problems, the path integral is not as easy to use as the Schrödinger equation, and most of the results obtained with it can be obtained more easily by other means. Nevertheless, one cannot help but be impressed with the elegance and beauty of the Feynman path integral, or recognize that it is a result of fundamental importance. Since in this course we are attempting to cover a variety of modern topics, we will study the Feynman path integral and some of its applications.

Standard references on path integrals include the books *Quantum Mechanics and Path Integrals* by Feynman and Hibbs and *Techniques and Applications of Path Integration* by L. S. Schulman. The first chapter of Feynman and Hibbs is especially recommended to those who wish to see a beautiful example of Feynman’s physical insight, and how it is applied to path integration.

For simplicity we will work (mostly) with a one-dimensional, time-independent system of the kinetic-plus-potential type, so that

\[ H = T + V = \frac{p^2}{2m} + V(x). \]  

(8.1)

The Feynman path integral is easily generalized to higher dimensions, and time-dependent potentials present no difficulty. With a little extra effort, magnetic fields can be incorporated. But there is greater difficulty in incorporating operators of a more general functional form, such as those with fourth powers of the momentum.

In any case, we write the propagator of the system as

\[ K(x, x_0, t) = \langle x | U(t) | x_0 \rangle, \]  

(8.2)
where we work strictly with positive times and suppress the \( \Theta(t) \) factor. Here of course \( U(t) = e^{-iHt/\hbar} \). We break the time interval \([0, t]\) up into a potentially large number \( N \) of small intervals of duration \( \epsilon \),

\[
\epsilon = \frac{t}{N}, \quad (8.3)
\]

so that

\[
U(t) = [U(\epsilon)]^N. \quad (8.4)
\]

The time evolution operator for time \( \epsilon \) is

\[
U(\epsilon) = e^{-i\epsilon(T+V)/\hbar}. \quad (8.5)
\]

Because the kinetic energy \( T \) and potential energy \( V \) do not commute, \( U(\epsilon) \) is not exactly equal to \( e^{-iT\hbar}e^{-iV\hbar} \). But since \( \epsilon \) is small, such a factorization is approximately correct, as we see by expanding in Taylor series:

\[
U(\epsilon) = 1 - \frac{i\epsilon}{\hbar}(T + V) + O(\epsilon^2) = e^{-iT\hbar}e^{-iV\hbar} + O(\epsilon^2). \quad (8.6)
\]

The epsilon ordering is not uniform, and the mathematics here is unusually unrigorous, even by physics standards. In fact, it is not easy to make the Feynman path integral mathematically rigorous, and for most practicing people the use of it requires some experience and intuition. In any case, since \( O(\epsilon^2) \) is the same as \( O(1/N^2) \), we can raise both sides of Eq. (8.6) to the \( N \)-th power to obtain

\[
U(t) = [e^{-iT\hbar}e^{-iV\hbar}]^N + O(1/N). \quad (8.7)
\]

Therefore we can write

\[
K(x, x_0, t) = \lim_{N \to \infty} \langle x | [e^{-iT\hbar}e^{-iV\hbar}]^N | x_0 \rangle. \quad (8.8)
\]

There are \( N \) factors here, so we can put \( N - 1 \) resolutions of the identity between them. We write the result in the form,

\[
K(x, x_0, t) = \lim_{N \to \infty} \int dx_1 \ldots dx_{N-1}
\]

\[
\times \langle x_N | e^{-iT\hbar}e^{-iV\hbar} | x_{N-1} \rangle \langle x_{N-1} | \ldots | x_1 \rangle \langle x_1 | e^{-iT\hbar}e^{-iV\hbar} | x_0 \rangle, \quad (8.9)
\]

where we have set

\[
x = x_N, \quad (8.10)
\]

in order to achieve greater symmetry in the use of subscripts. The kinetic energy is a function only of \( p \) and the potential only of \( x \), so we can bring the integrand into a form
we can evaluate explicitly by inserting $N$ more resolutions of the identity, this time using a momentum basis, one in the middle of each of the $N$ factors. This gives

$$K(x, x_0, t) = \lim_{N \to \infty} \int dx_1 \ldots dx_{N-1} dp_1 \ldots dp_N \times \langle x_N | e^{-i T/\hbar} | p_N \rangle \langle p_N | e^{-i V/\hbar} | x_{N-1} \rangle \ldots \ldots \langle x_1 | e^{-i T/\hbar} | p_1 \rangle \langle p_1 | e^{-i V/\hbar} | x_0 \rangle.$$ \hspace{1cm} (8.11)

Now the matrix elements can be evaluated. For example, in the $j$-th factor, we find

$$\langle x_j | e^{-i T/\hbar} | p_j \rangle = \frac{1}{\sqrt{2\pi \hbar}} \exp \left[ \frac{i}{\hbar} \left( -\epsilon p_j^2/2m + p_j x_j \right) \right],$$ \hspace{1cm} (8.12)

and

$$\langle p_j | e^{-i V/\hbar} | x_{j-1} \rangle = \frac{1}{\sqrt{2\pi \hbar}} \exp \left[ \frac{i}{\hbar} \left( -\epsilon V(x_{j-1}) - p_j x_{j-1} \right) \right].$$ \hspace{1cm} (8.13)

Therefore the whole integrand becomes a product of exponentials, and we can write

$$K(x, x_0, t) = \lim_{N \to \infty} \int dx_1 \ldots dx_{N-1} dp_1 \ldots dp_N \frac{1}{(2\pi \hbar)^N} \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[p_j (x_j - x_{j-1}) - \epsilon \left( \frac{p_j^2}{2m} + V(x_{j-1}) \right) \right] \right\}.$$ \hspace{1cm} (8.14)

This is a discretized version of the path integral in phase space. Notice that there is one more momentum integration than $x$-integration. We will not have much to say about the path integral in this form, except to notice that the exponent is $(i/\hbar)$ times a discretized version of an integral which is familiar in classical mechanics, namely,

$$A = \int p \, dx - H \, dt,$$ \hspace{1cm} (8.15)

as we can see if we identity $dx$ with $x_j - x_{j-1}$ and $dt$ with $\epsilon$. The quantity $A$ is the action associated with a path (sometimes a physical trajectory, sometimes not).

Instead, we will do the momentum integrations in Eq. (8.14). The $j$-th such integral is

$$\int dp_j \exp \left\{ \frac{i}{\hbar} \left[ p_j (x_j - x_{j-1}) - \frac{\epsilon p_j^2}{2m} \right] \right\} = \sqrt{\frac{2\pi \hbar m}{\epsilon}} \exp \left[ \frac{i}{\hbar} \frac{m (x_j - x_{j-1})^2}{2\epsilon} \right].$$ \hspace{1cm} (8.16)

Therefore the path integral (8.14) becomes

$$K(x_0, x, t) = \lim_{N \to \infty} \left( \frac{m}{2\pi i \hbar \epsilon} \right)^{N/2} \int dx_1 \ldots dx_{N-1} \times \exp \left\{ \frac{i \epsilon}{\hbar} \sum_{j=1}^{N} \left[ \frac{m (x_j - x_{j-1})^2}{2\epsilon} - V(x_{j-1}) \right] \right\}.$$ \hspace{1cm} (8.17)
This is a discretized version of the Feynman path integral in configuration space.

To visualize the integrations being performed in this integral, we notice that \( x_0 \) and \( x_N = x \) are fixed parameters of the integral, being the \( x \)-values upon which \( K \) depends, whereas all the other \( x \)'s, \( x_1, \ldots, x_{N-1} \), are variables of integration. Therefore we identify the sequence of numbers, \((x_0, x_1, \ldots, x_N)\) with a discretized version of a path \( x(t) \) in configuration space with fixed endpoints \((x_0, x_N = x)\), but with all intermediate points being variables. We think of the path \( x(t) \) as passing through the point \( x_j \) at time \( t_j = j \epsilon \), so that \( t_0 = 0 \) and \( t_N = t \). Then as \( N \to \infty \), we obtain a continuous representation of the path, and the integral turns into an integral over an infinite space of paths \( x(t) \) in configuration space, which are constrained to satisfy given endpoints at given endtimes. We notice also that the exponent is \((i/\hbar)\) times a discretized or (apparently) Riemann sum version of the classical action integral,

\[
A[x(t)] = \int_0^t dt L(x(t), \dot{x}(t)), \tag{8.18}
\]

where \( L \) is the classical Lagrangian,

\[
L(x, \dot{x}) = \frac{m \dot{x}^2}{2} - V(x). \tag{8.19}
\]

These observations motivate the compact notation for the path integral,

\[
K(x, x_0, t) = C \int d[x(t)] \exp \left( \frac{i}{\hbar} \int_0^t L dt \right), \tag{8.20}
\]

where \( C \) is the normalization constant seen explicitly in the discretized form (8.17), and where \( d[x(t)] \) represents the “volume” element in the infinite-dimensional path space. We do not let it bother us that the normalization constant \( C \) is really infinite.

Knowledge of the propagator \( K \) is equivalent to knowledge of the complete solution of the Schrödinger equation, both in its time-dependent and time-independent versions, for once \( K \) is known, we can Fourier transform it in time to obtain the energy-dependent Green’s function and from that the energy eigenvalues and eigenfunctions. Therefore the path integral constitutes a complete formulation of quantum mechanics which is alternative to the usual approach based on the Schrödinger equation. Although the path integral is usually more difficult to use computationally than the Schrödinger equation, its close connection with classical mechanics (through the use of the classical Lagrangian) is one of its most interesting features. You should also note the simple manner in which \( \hbar \) enters into the path integral: it serves merely as the unit of action, which makes the exponent dimensionless.
The path integral treats all paths connecting the given endpoints at the given endtimes in a democratic manner. For if we regard the square of the integrand as an (unnormalized) probability for a given path, then we see that this probability is the same for all paths, namely unity, since the integrand is just a phase factor. This probabilistic interpretation of paths is in fact correct, as explained in the first chapter of Feynman and Hibbs. Therefore all the subtle features of the path integral come from the interference of the probability amplitudes associated with different paths.

Since only the endpoints \(x_0\) and \(x_N = x\) are fixed in the path integral \((8.17)\), and since all intermediate points are variables of integration, it is clear that the paths which contribute to the path integral include some which are very strange looking. To visualize this, let us take the discretized version of the path integral and hold all the variables of integration fixed except one, say, \(x_j\). If we write \(\Delta x = x_j - x_{j-1}\), then during the integration over \(x_j\), \(\Delta x\) takes on all values from \(-\infty\) to \(+\infty\). This suggests that most of the paths \(x(t)\) which go into the path integral are not even continuous, since in time interval \(\Delta t = \epsilon\) any arbitrarily large value of \(\Delta x\) is allowed. But this conclusion is too drastic, and in a sense is not really correct. For as we will see later, it is appropriate to regard only those paths for which \(\Delta x \sim \sqrt{\Delta t}\) as contributing to the path integral, in spite of the fact that the absolute value of the integrand (namely unity) does not go to zero as \(\Delta x\) gets large. (Instead, this integrand oscillates itself to death as \(\Delta x\) gets large). In this interpretation, we see that the paths which contribute to the path integral are indeed continuous, for if \(\Delta x \sim \sqrt{\Delta t}\), then as \(\Delta t \to 0\), we also have \(\Delta x \to 0\). On the other hand, most of these paths are not differentiable, for we have \(\Delta x / \Delta t \sim (\Delta t)^{-1/2}\) as \(\Delta t \to 0\). Thus, a typical path contributing to the Feynman path integral is continuous everywhere but differentiable nowhere, and in fact has infinite velocity almost everywhere. To visualize such paths you may think of white noise on an oscilloscope trace, or a random walk such as Brownian motion in the limit in which the step size goes to zero. As you no doubt know, random walks also lead to the rule \(\Delta x \sim (\Delta t)^{1/2}\). Indeed path integrals of a different form (the so-called Wiener integral, with real, damped exponents instead of oscillating exponentials) are important in the theory of Brownian motion and similar statistical processes.

If typical paths in the path integral are not differentiable, i.e., if in some sense they have infinite velocity everywhere, then what is the meaning of the kinetic energy term in the Lagrangian in Eq. \((8.18)\)? The answer is that there is indeed an interpretational problem in the evaluation the action integral for such paths, and for this reason the compact notation \((8.20)\) for the path integral glosses over some things that are dealt with more properly in the discretized version \((8.17)\). In physical applications this discretized version is meaningful.
and the limit can be taken; one must resort to this procedure when questions concerning the differentiability of paths arise. Furthermore, the discretized version of the action integral in (8.17) is not really a Riemann sum approximation to a classical action integral, because in classical mechanics we (almost) always deal with paths which are differentiable. Instead, the integral is of a different type (an Ito integral).

Of course, the path integral includes as special cases the paths which are differentiable, but these are exceptional (they have measure zero in path space). Nevertheless, we know that the classical paths are differentiable. How do these show their privileged status in the classical limit, i.e., the limit in which $\hbar$ is small in comparison to typical actions of a problem?

The answer lies in the principle of stationary phase, which says that in an integral with a rapidly oscillating integrand, the principal contributions come from regions of the variable of integration which surround points where the phase is stationary, i.e., where it has only second order variations under first order variations in the variable of integration. In the Feynman path integral, the variable of integration is a path, so the stationary phase "points" are the paths of path space which satisfy

$$\frac{\delta A}{\delta x(t)} = 0. \quad (8.21)$$

But this is precisely the condition that $x(t)$ should be a classical path, according to Hamilton’s principle. A careful statement of Hamilton’s principle is made in the attached notes on Lagrangians and variational principles, and you should read those notes for more information on Hamilton’s principle and the variational formulations of classical mechanics.

Let us pursue the connection between the Feynman path integral and classical mechanics, which will lead us to semiclassical approximations for the propagator, as well as exact expressions in special cases. In the following discussion we will call on certain facts concerning the variational formulation of classical mechanics. These facts are developed more fully in the accompanying notes on variational principles and Lagrangian mechanics.

The variational formulation of classical mechanics begins with a pair of endpoints and endtimes, say, $(x_0, t_0)$ and $(x_1, t_1)$. In terms of these, we define a space of paths, which is the set

$$\{x(t)|x(t_0) = x_0, x(t_1) = x_1\}, \quad (8.22)$$

i.e., it is the set of all paths which join the two endpoints at the two endtimes. In classical mechanics, we want these paths to be continuous and differentiable (at least). This space of paths may contain actual classical motions, i.e., solutions of the equations of motion, but
most paths are not classically allowable motions. Next we invoke the Lagrangian function, 
$L(x, \dot{x}, t)$, which is an ordinary function of three variables. In the applications we have in
mind, this has the simple kinetic-minus-potential form (although we may consider magnetic
fields later). Next, we define the action functional,

$$A[x(t)] = \int_{t_0}^{t_1} dt \, L(x(t), \dot{x}(t), t),$$

which is defined for all paths in the space (8.22) (not just the ones which are classical
motions). Then, in accordance with Hamilton’s principle, we assert that the paths in the
space (8.22) which are solutions of the classical equations of motion are those for which the
action is stationary, as indicated by Eq. (8.21). In other words, for such paths, first order
variations in the path cause only second order variation in the action. Then, in a well known
manner, we show that Eq. (8.21) is equivalent to the usual Euler-Lagrange equations,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}.$$

Two common misconceptions concerning the variational formulation of classical me-
chanics, both of which are repeated in Sakurai, are the following. First, given the endpoints
and endtimes, the classical path joining them is not necessarily unique, nor does it necessarily
exist. That is, a path space such as (8.22), for given endpoints and endtimes and given
Lagrangian, may contain any number of classically allowable motions, from zero to infinity.
The reason for this is that the path space is specified by the initial and final positions of an
orbit, which do not necessarily specify a unique orbit. (If we were to specify the initial po-
sition and velocity, then of course the orbit would be unique.) Second, a classically allowed
path does not in general cause the action to be a minimum, but only stationary. In many
cases, the classically allowed path is actually a saddle point of the action functional, i.e., $A$
increases in some directions in the infinite dimensional path space, and decreases in others,
as we move away from a classically allowed path.

The physically allowed classical paths $x(t)$ are the solutions of the Euler-Lagrange
equations which belong to the path space (8.22). These can be thought of as being pa-
rameterized by the endpoints and endtimes, i.e., by $(x_0, t_0, x_1, t_1)$. Since there in general
there may be more than one such path, we may want to add a branch index $b = 1, 2, \ldots$ to
indicate which path we are talking about. Given such a path, it is of interest to evaluate
the action functional along it. The value of the action thereupon becomes a function of the
endpoints and endtimes, and the branch $b$. This function is Hamilton’s principal function,
and it is defined by

$$S = S_b(x_0, t_0, x_1, t_1) = A[x(t)],$$

(8.25)
where \( x(t) \) is the \( b \)-th physical path in the space (8.22). It is shown in the accompanying notes on variational principles that \( S \) satisfies the relations,

\[
\begin{align*}
\frac{\partial S}{\partial x_1} &= p_1, & \frac{\partial S}{\partial t_1} &= -H_1, \\
\frac{\partial S}{\partial x_0} &= -p_0, & \frac{\partial S}{\partial t_0} &= H_0, 
\end{align*}
\]

(8.26)

where \( p_0, p_1 \) and \( H_0, H_1 \) are respectively the momentum and Hamiltonian at the two endpoints. If the Lagrangian is time-independent, then energy is conserved, \( H_0 = H_1 \), and \( S \) is a function only of the elapsed time \( t_1 - t_0 \). In this case we often set \( t_0 = 0 \) and write simply \( S(x_1, x_0, t_1) \).

Let us return to the question of whether the action is minimum along a classical orbit. Of course, insofar as classical mechanics is concerned, we do not care whether the action is minimum, maximum or something else, because the equations of motion only depend on the vanishing of the first functional derivative of the action, as in Eq. (8.21). But it turns out, as we will see, that the phase of the propagator does depend on the nature of the stationary point of the action (in particular, it depends on the number of directions in path space for which the action functional is decreasing). Therefore it is of interest to examine the behavior of the action in the neighborhood of a classical path. We do this by replacing \( x(t) \) in Eq. (8.23) by \( x(t) + \delta x(t) \), where \( x(t) \) is a physical path and where \( \delta x(t_0) = \delta x(t_1) = 0 \). (We require this latter condition so that the modified path \( x(t) + \delta x(t) \) will belong to the path space (8.22).) Then we have a functional Taylor series,

\[
A[x(t) + \delta x(t)] = A[x(t)] + \int dt' \frac{\delta A}{\delta x(t')} \delta x(t') + \frac{1}{2} \int dt' dt'' \frac{\delta^2 A}{\delta x(t') \delta x(t'')} \delta x(t') \delta x(t'') + \ldots. \tag{8.27}
\]

The constant term on the right hand side is Hamilton’s principal function, the linear term vanishes on account of Hamilton’s principle (8.21), and the quadratic term gives the first nonvanishing correction to the action.

The quadratic term can be thought of as a the matrix element of a “wave function” \( \delta x(t) \), defined on the interval \( t_0 \leq t \leq t_1 \), sandwiched around an “operator” \( K \) whose time-domain kernel is

\[
K(t', t'') = \frac{1}{2} \frac{\delta^2 A}{\delta x(t') \delta x(t'')}, \tag{8.28}
\]

and a suggestive notation for this term is

\[
\langle \delta x | K | \delta x \rangle. \tag{8.29}
\]
We see that $K(t', t'')$ is real and symmetric, and therefore Hermitian. If the operator $K$ has only positive eigenvalues, then the first nonvanishing correction to the action in Eq. (8.27) is positive for all (nonvanishing) variations $\delta x(t)$, and the action is truly a minimum. But if some eigenvalues are positive and some negative, then the classical path is a saddle point of the action functional in path space. As we will see, we will be interested in the eigenvalues of $K$ when we make the stationary phase approximation on the Feynman path integral.

Next let us examine the mathematics of the stationary phase approximation. We begin with a 1-dimensional integral of the form,

$$\int dx \, e^{i\varphi(x)/\kappa}, \quad (8.30)$$

where $\kappa$ is a parameter. We wish to examine the behavior of this integral when $\kappa$ is small. Under these circumstances, the phase is rapidly varying, i.e., it takes only a small change $\Delta x$ (of order $\kappa$) to bring about a change of $2\pi$ in the overall phase. Therefore the rapid oscillations in the integrand nearly cancel one another, and the result is small. But an exception occurs around points $\bar{x}$ at which the phase is stationary, i.e., points $\bar{x}$ which are roots of

$$\frac{d\varphi}{dx}(\bar{x}) = 0. \quad (8.31)$$

For in the neighborhood of such points, the integrand is in phase for a larger $x$-interval (of order $\Delta x \sim \kappa^{1/2}$) than elsewhere, and furthermore there is one lobe of the oscillating integrand which is not cancelled by its neighbors. Therefore we obtain a good approximation if we expand the integrand about the stationary phase point,

$$\varphi(x) = \varphi(\bar{x}) + y\varphi'(\bar{x}) + \frac{y^2}{2}\varphi''(\bar{x}) + \ldots, \quad (8.32)$$

where $y = x - \bar{x}$ and where the linear correction term on the right hand side vanishes because of Eq. (8.31). Therefore the integral becomes a Gaussian with purely imaginary exponent, which can be done:

$$\int dx \, e^{i\varphi(x)/\kappa} = e^{i\varphi(\bar{x})/\kappa} \int dy \, e^{iy^2\varphi''(\bar{x})/2\kappa} = \sqrt{\frac{2\pi i \kappa}{\varphi''(\bar{x})}} \, e^{i\varphi(\bar{x})/\kappa}. \quad (8.33)$$

This result is valid if $\varphi''(\bar{x})$ is not too small. If $\varphi''(\bar{x})$ is very small or if it vanishes, then one must go to cubic order in the expansion (8.32) (a possibility we will not worry about).

The square root in Eq. (8.33) involves complex numbers, and the notation does not make the phase of the answer totally clear. The following notation is better:

$$\int dx \, e^{i\varphi(x)/\kappa} = e^{i\varphi(\bar{x})} \sqrt{\frac{2\pi i \kappa}{|\varphi''(\bar{x})|}} \, e^{i\varphi(\bar{x})/\kappa}, \quad (8.34)$$
where 
\[ \nu = \text{sgn} \varphi''(\bar{x}). \] (8.35)

Finally, we must acknowledge the possibility that there might be more than one stationary phase point (the roots of Eq. (8.31), which in general is a nonlinear equation). Indexing these roots by a branch index \( b \), the final answer is then a sum over branches,

\[ \int dx \, e^{i\varphi(x)/\kappa} = \sum_b e^{i\nu_b \pi/4} \sqrt{\frac{2\pi\kappa}{|\varphi''(\bar{x}_b)|}} e^{i\varphi(\bar{x}_b)/\kappa}. \] (8.36)

This is the stationary phase approximation for one-dimensional integrals.

Let us generalize this to multidimensional integrals. For this case we write

\[ x = (x_1, \ldots, x_n), \] (8.37)

without using vector (bold face) notation for the multidimensional variable \( x \). We consider the integral,

\[ \int d^n x \, e^{i\varphi(x)/\kappa}. \] (8.38)

As before, we define the stationary phase points \( \bar{x} \) as the roots of

\[ \frac{\partial \varphi}{\partial x_i}(\bar{x}) = 0, \] (8.39)

i.e., places where the gradient of \( \varphi \) vanish. (These are called critical points in mathematics.) Then we expand \( \varphi \) about a stationary phase point,

\[ \varphi(x) = \varphi(\bar{x}) + \frac{1}{2} \sum_{k\ell} y_k y_\ell \frac{\partial^2 \varphi(\bar{x})}{\partial x_k \partial x_\ell}, \] (8.40)

where we drop the vanishing linear terms and where \( y = x - \bar{x} \). Then the integral (8.38) becomes a multidimensional imaginary terms and where \( y = x - \bar{x} \). Then the integral (8.38) becomes a multidimensional imaginary Gaussian integral,

\[ \int d^n x \, e^{i\varphi(x)/\kappa} = e^{i\varphi(\bar{x})/\kappa} \int d^n y \, \exp \left( \frac{i}{2\kappa} \sum_{k\ell} y_k y_\ell \frac{\partial^2 \varphi(\bar{x})}{\partial x_k \partial x_\ell} \right). \] (8.41)

We do this by performing an orthogonal transformation \( z = Ry \), where \( R \) is an orthogonal matrix which diagonalizes \( \partial^2 \varphi/\partial x_k \partial x_\ell \). Since \( \det R = 1 \), we have \( d^n y = d^n z \). Then the integral becomes

\[ \int d^n x \, e^{i\varphi(x)/\kappa} = e^{i\varphi(\bar{x})/\kappa} \int d^n z \, \exp \left( \frac{i}{2\kappa} \sum_k \lambda_k z_k^2 \right), \] (8.42)
where \( \lambda_k \) are the eigenvalues of \( \partial^2 \varphi / \partial x_k \partial x_\ell \). This is a product of 1-dimensional Gaussian integrals which can be done as in Eq. (8.33). The result is

\[
\int d^n x e^{i \varphi(x)/\kappa} = e^{i \nu \pi / 4} (2\pi \kappa)^{n/2} \left| \det \frac{\partial^2 \varphi(\bar{x})}{\partial x_k \partial x_\ell} \right|^{-1/2} e^{i \varphi(\bar{x})/\kappa},
\]

where

\[
\nu = \nu_+ - \nu_-, \tag{8.44}
\]

where \( \nu_\pm \) is the number of \( \pm \) eigenvalues of \( \partial^2 \varphi / \partial x_k \partial x_\ell \). Finally, if there is more than one stationary phase point, we sum over them to obtain

\[
\int d^n x e^{i \varphi(x)/\kappa} = \sum_b e^{i \nu_b \pi / 4} (2\pi \kappa)^{n/2} \left| \det \frac{\partial^2 \varphi(\bar{x}_b)}{\partial x_k \partial x_\ell} \right|^{-1/2} e^{i \varphi(\bar{x}_b)/\kappa}. \tag{8.45}
\]

This is the stationary phase approximation for multidimensional integrals.

Now we return to the discretized version of the Feynman path integral, Eq. (8.17), and perform the stationary phase approximation on it. In comparison with the mathematical notes above, we set

\[
k = \frac{\hbar \epsilon}{m}, \tag{8.46}
\]

and we write

\[
\varphi(x_0, x_1, \ldots, x_N) = \sum_{j=1}^N \left[ \frac{1}{2} (x_j - x_{j-1})^2 - \frac{\epsilon^2}{m} V(x_{j-1}) \right], \tag{8.47}
\]

so that the integrand of Eq. (8.17) is \( e^{i \varphi/\kappa} \). We remember that in the path integral, \( x_0 \) and \( x_N = x \) are fixed parameters, and \( (x_1, \ldots, x_{N-1}) \) are the variables of integration. We differentiate \( \varphi \) twice, finding

\[
\frac{\partial \varphi}{\partial x_k} = 2x_k - x_{k+1} - x_{k-1} - \frac{\epsilon^2}{m} V'(x_k), \tag{8.48}
\]

and

\[
\frac{\partial^2 \varphi}{\partial x_k \partial x_\ell} = 2\delta_{k\ell} - \delta_{k+1,\ell} - \delta_{k-1,\ell} - \frac{\epsilon^2}{m} V''(x_k) \delta_{k\ell}. \tag{8.49}
\]

In these formulas, \( k, \ell = 1, \ldots, N - 1 \).

The stationary phase points are the discretized paths \( \bar{x}_k \) which make \( \partial \varphi / \partial x_k = 0 \). By Eq. (8.48), these satisfy

\[
\frac{\bar{x}_{k+1} - 2\bar{x}_k + \bar{x}_{k-1}}{\epsilon^2} = -\frac{1}{m} V'(\bar{x}_k), \tag{8.50}
\]

which is a discretized version of Newton’s laws,

\[
\frac{d^2 \bar{x}(\tau)}{d\tau^2} = -\frac{1}{m} V'(\bar{x}), \tag{8.51}
\]
where $x(\tau)$ is a classical path, satisfying $x(0) = x_0$, $x(t) = x$. (We use $\tau$ for the variable time upon which $x$ depends to distinguish it from $t$, the final time in the path integral.) In deriving Eqs. (8.50) and (8.51), we have effectively carried out a discretized version of the usual demonstration that Hamilton’s principle (8.21) is equivalent to Newton’s laws. These results show that

$$\lim_{N \to \infty} \frac{i m \varphi(\bar{x})}{\epsilon h} = \frac{i}{\hbar} S(x, x_0, t),$$

where Hamilton’s principal function $S$ is evaluated along the classical path $\bar{x}(\tau)$ which is the solution of Eq. (8.51).

To get the prefactor in the multidimensional stationary phase formula (8.45), we will need the determinant and the signs of the eigenvalues of $\frac{\partial^2 \varphi}{\partial x_k \partial x_\ell}$, evaluated on the stationary phase path $\bar{x}_k$ or (in the limit) $\bar{x}(\tau)$. (Interestingly enough, we do not need the actual eigenvalues themselves.) Let us work first on the determinant. Using Eq. (8.49) and setting

$$c_k = \frac{2}{m} V''(x_k),$$

the matrix in question is

$$\frac{\partial^2 \varphi}{\partial x_k \partial x_\ell} = \begin{pmatrix} 2 - c_1 & -1 & 0 & 0 & \cdots \\ -1 & 2 - c_2 & -1 & 0 & \cdots \\ 0 & -1 & 2 - c_3 & -1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{8.54}$$

This is a $(N - 1) \times (N - 1)$, tridiagonal matrix. To evaluate the determinant, we define $D_k$ as the determinant of the upper $k \times k$ diagonal block, and we define $D_0 = 1$. Then by Cramer’s rule, we find the recursion relation,

$$D_{k+1} = (2 - c_{k+1})D_k - D_{k-1}. \tag{8.55}$$

At this point we skip some details, and assert that this recursion relation can be used to show that

$$\lim_{N \to \infty} \det \frac{\partial^2 \varphi}{\partial x_k \partial x_\ell} = -\frac{m}{\epsilon} \left( \frac{\partial^2 S}{\partial x \partial x_0} \right)^{-1}. \tag{8.56}$$

Thus, both the phase of the stationary phase approximation, seen in Eq. (8.52), and the magnitude of the prefactor can be expressed in terms of Hamilton’s principal function along the classical orbit, $S(x, x_0, t)$.

For the overall phase of the stationary phase approximation to the Feynman path integral, however, we need in addition the signs of the eigenvalues of the matrix $\frac{\partial^2 \varphi}{\partial x_k \partial x_\ell}$. 

More exactly, as it turns out we only need the number of negative eigenvalues. For the phase of the prefactor in Eq. (8.43), we need the number of both positive and negative eigenvalues, as in Eq. (8.44), but since an \((N - 1) \times (N - 1)\) matrix has \((N - 1)\) eigenvalues, we have

\[
\nu_+ + \nu_- = N - 1, \tag{8.57}
\]

so that

\[
\nu = N - 1 - 2\nu_. \tag{8.58}
\]

But the prefactor of the discretized Feynman path integral in Eq. (8.17) has an phase of its own, \(e^{-iN\pi/4}\), so the overall phase of the prefactor in the stationary phase approximation is

\[
e^{-iN\pi/4} e^{i\nu\pi/4} = e^{-i\pi/4} e^{-i\mu\pi/2}, \tag{8.59}
\]

where \(\mu = \nu_-.\) It turns out that the number of negative eigenvalues \(\mu\) approaches a definite limit as \(N \to \infty\), so the overall phase of path integral also approaches a definite limit (as we expect). Similarly, we find that the magnitude of the prefactor also approaches a definite limit as \(N \to \infty\) (all the \(\epsilon\)'s cancel).

We may now gather all the pieces of the stationary phase approximation together. We find

\[
K(x, x_0, t) = \frac{e^{-i\mu\pi/2}}{\sqrt{2\pi i\hbar}} \left| \frac{\partial^2 S}{\partial x \partial x_0} \right|^{1/2} \exp \left[ \frac{i}{\hbar} S(x, x_0, t) \right]. \tag{8.60}
\]

This is in the case of a single classical path connecting the endpoints and endtimes. If there is more than one such path, we sum over the contributions,

\[
K(x, x_0, t) = \sum_b \frac{e^{-i\mu_0\pi/2}}{\sqrt{2\pi i\hbar}} \left| \frac{\partial^2 S_b}{\partial x \partial x_0} \right|^{1/2} \exp \left[ \frac{i}{\hbar} S_b(x, x_0, t) \right]. \tag{8.61}
\]

Equation (8.60) or (8.61) is the Van Vleck formula, and it is the WKB or semiclassical approximation to the propagator for 1-dimensional problems. The 3-dimensional formula is almost the same,

\[
K(x, x_0, t) = \sum_b \frac{e^{-i\mu_0\pi/2}}{(2\pi i\hbar)^{3/2}} \left| \det \frac{\partial^2 S_b}{\partial x \partial x_0} \right|^{1/2} \exp \left[ \frac{i}{\hbar} S_b(x, x_0, t) \right]. \tag{8.62}
\]

The Van Vleck formula amounts to approximating the Feynman path integral by including all classical paths connecting the given endpoints at the given endtimes, as well as the second order contributions coming from fluctuations about such paths. The derivation of the Van Vleck formula amounts to doing a huge Gaussian integration.
But if the potential energy is at most a quadratic polynomial in $x$, i.e., if it has the form

$$V(x) = a_0 + a_1 x + a_2 x^2, \quad (8.63)$$

a case which includes the free particle, the particle in a gravitational field, and the harmonic oscillator, then the stationary phase approximation is exact, since the exact exponent in the Feynman path integral is a quadratic function of the path. In such cases the Van Vleck formula is exact. It is also exact in certain other cases, such as the motion of a charged particle in a uniform magnetic field (in such cases the Lagrangian has a velocity dependence, but it is still a quadratic function of the path). Indeed, it is only in such cases that an exact evaluation of the path integral is at all easy to obtain.

Let us evaluate the Van Vleck formula for the case of a free particle in one dimension. Most of the calculation is classical. The Lagrangian is

$$L = \frac{m \dot{x}^2}{2}, \quad (8.64)$$

which of course is the kinetic energy which is conserved along the classical motion. Therefore Hamilton’s principal function is

$$S = \int L \, dt = \frac{m \dot{x}^2 t}{2}. \quad (8.65)$$

But it is customary to express $S$ as a function of the endpoints and endtimes, not the velocities, so we invoke the equation of the classical path,

$$x = x_0 + \dot{x}_0 t, \quad (8.66)$$

which we solve for $\dot{x}_0$,

$$\dot{x}_0 = \frac{x - x_0}{t}. \quad (8.67)$$

But since $\dot{x} = \dot{x}_0$ (the velocity is constant along the path), we can substitute Eq. (8.67) into Eq. (8.65) to obtain

$$S(x, x_0, t) = \frac{m(x - x_0)^2}{2t}. \quad (8.68)$$

Furthermore, we can see from Eq. (8.66) that the classical path connecting the endpoints and endtimes is unique (that is, given $(x, x_0, t)$, the initial velocity $\dot{x}_0$ is unique). Therefore there is only one term in the Van Vleck formula.

Finally, we need the number of negative eigenvalues $\mu$ of the second variation of the action. One can show that this number is always zero (the action is a minimum) in kinetic-plus-potential problems, if the time $t$ is short enough. But for the free particle, it turns
out that $\mu = 0$ for all times. To see this, we evaluate the action functional along the path $x(t) + \delta x(t)$, where $x(t)$ is the physical motion (8.66), and where $\delta x(t)$ is a nonphysical variation satisfying $\delta x(0) = \delta x(t) = 0$. Then we have

$$A[x(t) + \delta x(t)] = \int_0^t \frac{m}{2} [\dot{x}(\tau) + \delta \dot{x}(\tau)]^2 d\tau$$

$$= \int_0^t \frac{m}{2} \dot{x}(\tau)^2 d\tau + \int_0^t m \dot{x}(\tau) \delta \dot{x}(\tau) d\tau + \int_0^t \frac{m}{2} \delta \dot{x}(\tau)^2 d\tau$$

$$= S(x, x_0, t) + \int_0^t \frac{m}{2} \delta \dot{x}(\tau)^2 d\tau, \quad (8.69)$$

where the linear term vanishes in accordance with Hamilton’s principle. On the other hand, the quadratic term is obviously nonnegative for any choice of $\delta x(\tau)$, and vanishes only if $\delta x(\tau) = 0$. [The case $\delta x(\tau) = \text{const} \neq 0$ can be excluded because of the boundary conditions, $\delta x(0) = \delta x(t) = 0$.] Therefore all the eigenvalues of the second variation of the action are positive, and $\mu = 0$.

This argument is fine as far as it goes, but you may be interested to actually see the eigenvalues and eigenfunctions of the second variation of the action. To do this, we want to write the second variation in the form $\langle \delta x | K | \delta x \rangle$ for some operator $K$. We can do this by integrating by parts, to obtain

$$\int_0^t \frac{m}{2} \delta \dot{x}(\tau)^2 d\tau = \int_0^t d\tau \delta x(\tau) \left( -\frac{m}{2} \frac{d^2}{d\tau^2} \right) \dot{x}(\tau) = \langle \delta x | K | \delta x \rangle, \quad (8.70)$$

so that $K = -(m/2)d^2/d\tau^2$. Therefore the eigenvalue problem for the second variation of the action is

$$-\frac{m}{2} \frac{d^2 \xi_n(\tau)}{d\tau^2} = \kappa_n \xi_n(\tau), \quad (8.71)$$

where $\xi_n(\tau)$ are the eigenfunctions, satisfying $\xi_n(0) = \xi_n(t) = 0$, and where $\kappa_n$ are the eigenvalues. This has the same mathematical form of a quantum mechanical particle in a box, so the (unnormalized) eigenfunctions are

$$\xi_n(\tau) = \sin \left( \frac{n \pi \tau}{l} \right), \quad n = 1, 2, \ldots, \quad (8.72)$$

and the eigenvalues are

$$\kappa_n = \frac{m n^2 \pi^2}{2 l^2}. \quad (8.73)$$

These eigenvalues are all positive, as claimed.

We may now gather together all the pieces of the Van Vleck formula for the free particle. We find,

$$K(x, x_0, t) = \left( \frac{m}{2 \pi i \hbar} \right)^{1/2} \exp \left[ \frac{i m (x - x_0)^2}{2 \hbar t} \right]. \quad (8.74)$$
This is the same result as Eq. (7.17). You will appreciate that the derivation in Notes 7 was much easier; this is an example of what people mean when they say that the path integral is harder to use than the Schrödinger equation. Nevertheless, there are certain applications where the use of the path integral leads quickly to striking intuition and remarkable results.

As a final topic in our discussion of the Feynman path integral, we will provide an explicit demonstration that it is actually equivalent to the Schrödinger equation. It suffices to do this for infinitesimal times, since finite-time propagation can be built up out of infinitesimal increments. For variety we work in three dimensions.

Since the Schrödinger equation is

\[ i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right] \psi(x,t), \]  

if we propagate \( \psi \) from \( t = 0 \) to \( t = \epsilon \) we have

\[ \psi(x, \epsilon) = \psi(x, 0) - \frac{i\epsilon}{\hbar} \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right] \psi(x, 0) + O(\epsilon^2). \]  

To see how the same result comes out of the path integral, we invoke the propagator for time \( \epsilon \), and write

\[ \psi(x, \epsilon) = \int d^3y \ K(x, y, \epsilon) \psi(y, 0), \]  

where we use \( y \) instead of \( x_0 \). Now we invoke the 3-dimensional version of the discretized Feynman path integral [see Eq. (8.17)], which is

\[ K(x_0, x, t) = \lim_{N \to \infty} \left( \frac{m}{2\pi i\hbar \epsilon} \right)^{3N/2} \int d^3x_1 \ldots d^3x_{N-1} \]

\[ \times \exp \left\{ \frac{i\epsilon}{\hbar} \sum_{j=1}^{N} \left[ \frac{m(x_j - x_{j-1})^2}{2\epsilon^2} - V(x_{j-1}) \right] \right\}. \]  

But for time \( \epsilon \), we need only one of the factors in the discretized integrand, i.e., we set \( N = 1 \) and replace the kernel \( K(x, y, \epsilon) \) in Eq. (8.77) by

\[ K(x, y, \epsilon) = \left( \frac{m}{2\pi i\hbar \epsilon} \right)^{3/2} \exp \left\{ \frac{i\epsilon}{\hbar} \left[ \frac{m(x - y)^2}{2\epsilon^2} - V(y) \right] \right\}. \]  

Now we write \( y = x + \xi \), so that \( \xi \) is the displacement vector from the final position \( x \) to the initial position \( y \). Thus, we have

\[ \psi(x, \epsilon) = \left( \frac{m}{2\pi i\hbar \epsilon} \right)^{3/2} \int d^3\xi \]

\[ \times \exp \left[ \frac{im|\xi|^2}{2\epsilon h} - \frac{i\epsilon}{\hbar} V(x + \xi) \right] \psi(x + \xi, 0). \]  

(8.80)
We wish to expand this integral out to first order in $\epsilon$, in order to compare with Eq. (8.76). The key do doing this is to realize that the principal contributions to the integral come from regions where $\xi \sim O(\epsilon^{1/2})$, because of the rapidly oscillating factor $\exp(i m |\xi|^2/2\epsilon \hbar)$ in the integrand. That such regions are small when $\epsilon$ is small is gratifying, because it means that parts of the initial wave function $\psi(x,0)$ at one spatial position $x$ cannot influence parts of the final wave function $\psi(x,\epsilon)$ at distant points in arbitrarily short times. Therefore we expand the integrand of Eq. (8.80) out to $O(\epsilon)$, treating $\xi$ as $O(\epsilon^{1/2})$. According to this rule, we cannot expand the factor $\exp(i m |\xi|^2/2\epsilon \hbar)$, because the exponent is $O(1)$, but the exponent in the factor $\exp(-i e V/\hbar)$ is small and this factor can be expanded. Similarly, we can expand the wavefunction $\psi(x+\xi,0)$. Thus, the integral becomes

$$
\left(\frac{m}{2\pi i\hbar \epsilon}\right)^{3/2} \int d^3\xi \exp\left(\frac{i m |\xi|^2}{2\epsilon \hbar}\right) \left[1 - \frac{i \epsilon}{\hbar} V(x + \xi) + \ldots\right]
$$

$$
\times \left[\psi(x,0) + \xi \cdot \nabla \psi(x,0) + \frac{1}{2} \xi \xi : \nabla \nabla \psi(x,0) + \ldots\right],
$$

where

$$
\xi \xi : \nabla \nabla \psi = \sum_{ij} \xi_i \xi_j \frac{\partial^2 \psi}{\partial x_i \partial x_j}.
$$

Now we collect terms by orders of $\epsilon$. The term in the product of the two expansions in Eq. (8.81) which is $O(\epsilon^0)$ is simply $\psi(x,0)$, which is independent of $\xi$ and comes out of the integral. The rest of the integral just gives unity.

The term in the product of the expansions which is $O(\epsilon^{1/2})$ is $\xi \cdot \nabla \psi(x,0)$, but since this term is odd in $\xi$ and the rest of the integrand is even, this term integrates to zero. This is good, because we don’t want to see any fractional powers of $\epsilon$.

The $O(\epsilon)$ term in the product of the expansions is

$$
-\frac{i \epsilon}{\hbar} V(x) \psi(x,0) + \frac{1}{2} \xi \xi : \nabla \nabla \psi(x,0),
$$

where we drop the $\xi$ correction to the potential energy because that is really $O(\epsilon^{3/2})$. Now we invoke the integral,

$$
\left(\frac{a}{i\pi}\right)^{3/2} \int d^3\xi \exp(i a |\xi|^2) \xi_i \xi_j = \frac{i}{2a} \delta_{ij},
$$

which shows that the $O(\epsilon)$ term integrates into

$$
-\frac{i \epsilon}{\hbar} V(x) \psi(x,0) + \frac{i \epsilon \hbar}{2m} \nabla^2 \psi(x,0).
$$

Altogether, these results are equivalent to Eq. (8.76).