These notes contain an easier route to the derivation of the classical electromagnetic field Hamiltonian than that presented in Notes 32. The main difference is that in these notes we obtain the classical Hamiltonian directly from the equations of motion, instead of proceeding through the Lagrangian, as is done in Notes 32. These notes have much in common with Notes 27, but the notation has been changed slightly. This should not cause confusion, since Notes 27 were optional. The notation in these notes is consistent with that in Notes 32.

We begin with the free electromagnetic field ($\rho = 0$ and $J = 0$), for which Maxwell’s equations are

$$\begin{align*}
\nabla \cdot E &= 0, \\
\nabla \cdot B &= 0, \\
\nabla \times B &= \frac{1}{c} \frac{\partial E}{\partial t}, \\
\nabla \times E &= -\frac{1}{c} \frac{\partial B}{\partial t}.
\end{align*}$$

(34.1)

We will choose Coulomb gauge for the vector potential,

$$\nabla \cdot A = 0,$$

(34.2)

which implies that the scalar potential vanishes, $\phi = 0$. For various pedagogical reasons, Coulomb gauge is convenient for a first pass at quantizing the electromagnetic field, although it suffers from a lack of manifest relativistic covariance. It also causes certain quantities which appear in the theory (such as the longitudinal electric field) to propagate instantaneously from the source to the field point, although this does not violate causality in a physical sense because the physically measurable fields propagate causally (at the speed of light). On the other hand, Coulomb gauge is probably the best gauge to use for the interaction of low-velocity material systems, such as typical atoms, molecules, and nuclei, with radiation. The advantages and disadvantages of Coulomb gauge are discussed more fully in Notes 32. In any case, under our assumptions, the electric and magnetic fields are given in terms of the vector potential by

$$E = -\frac{1}{c} \frac{\partial A}{\partial t}, \quad B = \nabla \times A.$$ 

(34.3)

The vector potential of a plane light wave is

$$A(r, t) = C_0 \epsilon e^{i(k \cdot r - \omega t)} + \text{c.c.},$$

(34.4)
where $C_0$ is the complex amplitude, where $\epsilon$ is a unit polarization vector, and where $\omega = ck$. The polarization vector is orthogonal to the direction of propagation, $\epsilon \cdot k = 0$, and is allowed to be complex in order to give different phases to the different components of $A$. Although the vector potential $A(r,t)$ itself is real, $C_0$ is physically meaningful as a complex number, since its magnitude is the real amplitude of the wave and its phase is the overall phase of the wave. The electric and magnetic fields of the wave are easily obtained from Eq. (34.3).

It is a fact that an arbitrary solution to the free-field Maxwell equations (34.1) can be represented as a linear combination of real plane light waves of the form (34.4). This is obviously just a kind of Fourier expansion, but there are certain details to worry about since both $A(r,t)$ and the constituent plane waves are real and the vector field $A(r,t)$ is transverse. In the following we will make a somewhat careful analysis of this Fourier expansion, in order to obtain a set of real plane waves which is complete but not overcomplete. When we have done this, the complex mode amplitudes [the analogs of $C_0$ in Eq. (34.4)] will constitute a complete and minimal description of the dynamical state of the electromagnetic field. It will also turn out that there is one degree of freedom of the electromagnetic field for each constituent plane light wave.

We remark that in classical mechanics, what we mean by the “dynamical state” of a system is the information needed to specify a complete set of initial conditions, so that if the dynamical state is known at one time, then in principle it can be determined at any other time. Thus, in an ordinary mechanical system, a specification of the $q$’s and $\dot{q}$’s (or equivalently, the $q$’s and $p$’s) is a specification of the dynamical state. Specifying the $q$’s alone is not sufficient. Similarly, for the free electromagnetic field (with our gauge conventions), the dynamical state is specified by $A(r,t)$ and $\partial A(r,t)/\partial t$ at some instant of time [$A(r,t)$ alone is not sufficient]. We also remark that the “number of degrees of freedom” of a dynamical system is the number of $q$’s; the number of phase space variables ($q$’s and $p$’s) is twice the number of degrees of freedom.

To develop the desired basis of real plane waves, we begin by introducing periodic boundary conditions in a box of volume $V = L^3$. The conventions we use for this are described by Eqs. (32.67)–(32.72). Since the vector potential is now assumed to be spatially periodic at each instant of time, it can be expanded in an ordinary Fourier series,

$$ A(r,t) = \frac{1}{\sqrt{V}} \sum_k A_k(t) e^{ik \cdot r}, \quad (34.5) $$

which serves to define the Fourier coefficients $A_k$. These coefficients $A_k$ are complex vectors, one for each lattice point in $k$ space. If $(A_k, \tilde{A}_k)$ are known at each $k$ lattice point, then we have a complete specification of the dynamical state of the free electromagnetic field, since
they determine $\mathbf{A}(r,t)$ and $\partial \mathbf{A}(r,t)/\partial t$; however, the coefficients $(\mathbf{A}_k, \dot{\mathbf{A}}_k)$ satisfy certain constraints and are therefore not the most convenient way of specifying this dynamical state.

There are two constraints. First, since $\mathbf{A}(r,t) = \mathbf{A}(r,t)^*$ is real, the Fourier coefficients satisfy

$$
\mathbf{A}_k = \mathbf{A}^*_{-k}. \quad (34.6)
$$

The Fourier coefficients $\mathbf{A}_k$ themselves are complex, but the value of the coefficient at one lattice point in $k$-space is related to the value at the opposite lattice point by Eq. (34.6). Second, the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ implies

$$
k \cdot \mathbf{A}_k = 0, \quad (34.7)
$$

which is the condition that $\mathbf{A}$ be a transverse vector field. The subject of transverse and longitudinal vector fields is discussed in Notes 32, and we will call on that presentation freely. [See Eqs. (32.32)-(32.50).]

To find a more convenient way of specifying the dynamical state of the electromagnetic field, we will examine the equations of motion satisfied by $\mathbf{A}_k$. As for $\mathbf{A}(r,t)$, it satisfies the the wave equation,

$$
\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \quad (34.8)
$$
as follows from Eqs. (34.1), (34.2), and (34.3). This implies that the equation of motion for the Fourier coefficients is

$$\frac{d^2 \mathbf{A}_k}{dt^2} + \omega^2 \mathbf{A}_k = 0. \quad (34.9)$$

Here and throughout the following discussion, $\omega$ is always understood to be a function of $k$ or $k$, $\omega = \omega_k = c|k| = ck$. Equation (34.9) is that of a harmonic oscillator, or more exactly, a vector of complex harmonic oscillators. The general solution in a mathematical sense is

$$
\mathbf{A}_k(t) = \mathbf{C}_{0k}^{(+)} e^{-i\omega t} + \mathbf{C}_{0k}^{(-)} e^{i\omega t}, \quad (34.10)
$$

where the vectors $\mathbf{C}_{0k}^{(\pm)}$ are two constant, complex vectors which are the constants of integration. These vectors are closely related to the initial conditions [they are easily expressible in terms of $\mathbf{A}_k(0)$ and $\dot{\mathbf{A}}_k(0)$], which is the reason for the 0 subscript. The $\pm$ sign indicates which of the two terms $e^{\pm i\omega t}$ the coefficients are attached to; the term $e^{-i\omega t}$ is considered to be the positive frequency term.

However, the solutions we are interested in physically satisfy the constraints (34.6) and (34.7). The constraint (34.6) implies

$$
\mathbf{A}_k(t) = \mathbf{A}_{-k}^*(t) = \mathbf{C}_{0,-k}^{(+)} e^{i\omega t} + \mathbf{C}_{0,-k}^{(-)} e^{-i\omega t}, \quad (34.11)
$$
or, since $e^{-i\omega t}$ and $e^{i\omega t}$ are linearly independent,

$$C_{0k}^{(\pm)} = C_{0,-k}^{(\mp)*}. \quad (34.12)$$

This means that for the solutions $A_k(t)$ we are interested in, the vector $C_{0k}^{(+)}$ at one lattice site is related to the vector $C_{0k}^{(-)}$ at the opposite lattice site, so it suffices to specify only the $C_{0k}^{(+)}$ vectors at all lattice sites in $k$-space, whereupon the $C_{0k}^{(-)}$ vectors become known also. Let us therefore abbreviate the notation, and write

$$C_{0k}^{(+)} = C_{0k}, \quad C_{0k}^{(-)} = C_{0,-k}^{*}. \quad (34.13)$$

Then the solution (34.10) becomes

$$A_k(t) = C_{0k} e^{-i\omega t} + C_{0,-k}^{*} e^{i\omega t}, \quad (34.14)$$

and the Fourier series (34.5) becomes

$$A(r, t) = \frac{1}{\sqrt{V}} \sum_k \left[ C_{0k} e^{i(k \cdot r - \omega t)} + C_{0,-k}^{*} e^{i(k \cdot r + i\omega t)} \right]. \quad (34.15)$$

Now if we replace the dummy index $k$ in the second term by $-k$, we obtain a simpler form,

$$A(r, t) = \frac{1}{\sqrt{V}} \sum_k \left[ C_{0k} e^{i(k \cdot r - \omega t)} + c.c. \right]. \quad (34.16)$$

As for the second constraint (34.7), it implies that the constant vectors $C_{0k}$ are transverse,

$$k \cdot C_{0k} = 0. \quad (34.17)$$

Thus, for the solutions we are interested in, the vectors $C_{0k}$ only have two independent components at each point of $k$-space. We incorporate these constraints by introducing two polarization vectors $\epsilon_{k\mu}$, $\mu = 1, 2$, and writing

$$C_{0k} = \sum_{\mu=1}^{2} C_{0k\mu} \epsilon_{k\mu}. \quad (34.18)$$

Polarization vectors are defined and discussed in Notes 32 [see Eqs. (32.76)--(32.78)]. We also use the notation $\lambda = (k, \mu)$, and refer to $\lambda$ as a “mode index.” Then the expression (34.16) for the vector potential becomes

$$A(r, t) = \frac{1}{\sqrt{V}} \sum_{\lambda} \left[ C_{0\lambda} e^{i(k \cdot r - \omega t)} + c.c. \right]. \quad (34.19)$$
This is the desired representation of the state of the free electromagnetic field as a linear combination of real plane waves. It is a sum over the modes $\lambda$ (wave vector plus polarization) of the field, with the complex coefficients $C_{0\lambda}$ giving the amplitudes and phases of the waves in each mode.

Finally, let us write

$$C_k(t) = C_{0k} e^{-i\omega t}, \quad C_\lambda(t) = C_{0\lambda} e^{-i\omega t}, \quad (34.20)$$

to incorporate the time dependence into the complex mode amplitudes. This causes the mode amplitudes to satisfy the equation of motion,

$$\dot{C}_k = -i \omega C_k, \quad \dot{C}_\lambda = -i \omega C_\lambda, \quad (34.21)$$

and it allows us to write the vector potential in the form,

$$A(r, t) = \frac{1}{\sqrt{V}} \sum_k A_k e^{ik \cdot r} = \frac{1}{\sqrt{V}} \sum_k \left[ C_k(t) e^{ik \cdot r} + \text{c.c.} \right]$$

$$= \frac{1}{\sqrt{V}} \sum_\lambda \left[ C_\lambda(t) \epsilon_\lambda e^{ik \cdot r} + \text{c.c.} \right]. \quad (34.22)$$

As we have said, the dynamical state of the electromagnetic field is specified by $A(r, t)$ and $\partial A(r, t)/\partial t$, of which $A(r, t)$ is given in terms of the $C_k$ or $C_\lambda$ by Eq. (34.22). The time derivative $\partial A(r, t)/\partial t$ can also be expressed in terms of $C_k$ or $C_\lambda$, by differentiating Eq. (34.22) and using Eq. (34.21):

$$\frac{\partial A(r, t)}{\partial t} = \frac{1}{\sqrt{V}} \sum_k \dot{A}_k e^{ik \cdot r} = \frac{1}{\sqrt{V}} \sum_k \left[ -i \omega C_k(t) e^{ik \cdot r} + \text{c.c.} \right]$$

$$= \frac{1}{\sqrt{V}} \sum_\lambda \left[ -i \omega C_\lambda(t) \epsilon_\lambda e^{ik \cdot r} + \text{c.c.} \right]. \quad (34.23)$$

By equating coefficients of $e^{ik \cdot r}$ in Eqs. (34.22) and (34.23), we can find a relation between the Fourier coefficients $A_k$ and the $C_k$:

$$A_k = C_k + C^*_{-k},$$

$$\dot{A}_k = -i \omega (C_k - C^*_{-k}). \quad (34.24)$$

These can be solved,

$$C_k = \frac{1}{2} \left( A_k + \frac{i}{\omega} \dot{A}_k \right)$$

$$C^*_{-k} = \frac{1}{2} \left( A_k - \frac{i}{\omega} \dot{A}_k \right). \quad (34.25)$$
The second of Eqs. (34.25) is equivalent to the first, in view of the reality condition (34.6) on the $A_k$. Equations (34.24) and (34.25) show that the vectors $C_k$ contain the same information as in $(A_k, \dot{A}_k)$, and so constitute a complete specification of the dynamical state of the field. The transformation specified by Eqs. (34.24) or (34.25) can be regarded as a transformation on the phase space of the electromagnetic field.

Now we would like to find a classical Hamiltonian which will give the known equations of motion via Hamilton’s equations. The classical equations of motion are either (34.8) or (34.9), which themselves are equivalent (under our assumption of Coulomb gauge) to the free field Maxwell’s equations (34.1). The proper way to find this Hamiltonian is to proceed through a Lagrangian and the Legendre transformation as is done in Notes 32, but here we will take a shortcut. Since the Hamiltonian in classical mechanics is usually the energy of the system, we guess that this will be true for the field as well. The energy of the electromagnetic field is

$$H = \frac{1}{8\pi} \int d^3r (E^2 + B^2), \quad (34.26)$$

which we denote by $H$ in anticipation that this will prove to be the Hamiltonian. When we are thinking in terms of periodic boundary conditions, the integral should be carried out only over the volume of the box.

In order to use Hamilton’s equations in the ordinary sense, we must express the Hamiltonian as a function of some set of $q$’s and $p$’s. We do not yet know what the $q$’s and $p$’s are, but it will help to guess them if we express the Hamiltonian in terms of the coefficients $C_\lambda$. To do this, we first write out Fourier series for the electric and magnetic fields by differentiating Eq. (34.22) and using Eq. (34.3). For the time derivative of the coefficients $C_\lambda$, we use Eq. (34.21). Thus we find

$$E(r, t) = \frac{1}{\sqrt{V}} \sum_\lambda \left[ i\omega e^{i\mathbf{k} \cdot \mathbf{r}} + \text{c.c.} \right],$$

$$B(r, t) = \frac{1}{\sqrt{V}} \sum_\lambda \left[ iC_\lambda (\mathbf{k} \times \mathbf{\epsilon}_\lambda) e^{i\mathbf{k} \cdot \mathbf{r}} + \text{c.c.} \right]. \quad (34.27)$$

Let us look first at the integral of $E^2$ in the Hamiltonian (34.26). This is

$$E^2\text{-term} = \frac{1}{8\pi} \int_V d^3r \frac{1}{\sqrt{V}} \sum_{\lambda, \lambda'} \left[ i\omega e^{i\mathbf{k} \cdot \mathbf{r}'} - i\omega e^{i\mathbf{k} \cdot \mathbf{r}'} \right]$$

$$\cdot \left[ i\omega' e^{i\mathbf{k}' \cdot \mathbf{r}} - i\omega' e^{i\mathbf{k}' \cdot \mathbf{r}} \right], \quad (34.28)$$

where we integrate only over the volume of the box. There are four major terms in this
integral; let us look at one of the cross terms, namely,

\[
C_\lambda C_{\lambda'}^*-\text{term} = \frac{1}{8\pi} \int_V d^3r \frac{1}{V} \sum_{\lambda\lambda'} \frac{\omega_{\lambda'}}{c^2} C_\lambda C_{\lambda'}^* (\epsilon_\lambda \cdot \epsilon_{\lambda'}^*) e^{i(k-k') \cdot r} = \frac{1}{8\pi} \sum_{\lambda} \frac{\omega_{\lambda}}{c^2} C_\lambda C_{\lambda}^* (\epsilon_\lambda \cdot \epsilon_{\lambda}^*) \delta_{k,k'} = \frac{1}{8\pi} \sum_{k_{\mu}k'_{\mu'}} \omega_{k_{\mu}k'_{\mu'}}^2 |C_{k_{\mu}k'_{\mu'}}|^2 = \frac{1}{8\pi} \sum_{\lambda} \frac{\omega_{\lambda}^2}{c^2} |C_{\lambda}|^2 , \tag{34.29}
\]

where we have carried out the spatial integration in the second equality, written \( \lambda = (k,\mu) \) and \( \lambda' = (k',\mu') \) and done the \( k' \)-sum in the third equality, and then used Eq. (32.76a) in the fourth equality. Thus we have evaluated one of the cross terms. By a similar calculation, we find that the other cross term in \( E^2 \) integral, as well as the two cross terms in the \( B^2 \) integral, all give the same answer as Eq. (34.29). As for the other terms (the \( C_\lambda C_{\lambda}^* \)-terms and the \( C_\lambda^* C_{\lambda}^* \)-terms), we find that these cancel when the electric and magnetic contributions are added. Altogether, the Hamiltonian becomes

\[
H = \frac{1}{2\pi} \sum_{\lambda} \frac{\omega_{\lambda}^2}{c^2} |C_{\lambda}|^2 . \tag{34.30}
\]

Let us now return to the time evolution of the complex mode amplitudes \( C_{\lambda} \), given by Eq. (34.20). As viewed in the complex plane, the complex number \( C_{\lambda}(t) \) traces out a circle at frequency \( \omega \) in the clockwise direction. This reminds us of the evolution of a harmonic oscillator in the \( q-p \) phase plane, as discussed and illustrated in Notes 6 [see Eq. (6.39)], and it suggests that the real and imaginary parts of \( C_{\lambda} \) be treated as the \( q \) and \( p \) of a harmonic oscillator, say, \( C_{\lambda} = \text{const}(Q_{\lambda} + iP_{\lambda}) \). To find the constant of proportionality, we demand that the Hamiltonian \( H \) in Eq. (34.30) should emerge as a sum of harmonic oscillator Hamiltonians, one for each mode. The Hamiltonian of a one-dimensional mechanical harmonic oscillator is usually taken to be

\[
H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} , \tag{34.31}
\]

but by a simple canonical transformation [set \( p = (m\omega)^{1/2} p' \), \( q = (m\omega)^{-1/2} q' \), then drop the primes] it can be cast into the more symmetrical form,

\[
H = \frac{\omega}{2}(p^2 + q^2), \tag{34.32}
\]
whereupon the orbits are circles in phase space. We can also bring the field Hamiltonian (34.30) into this form if we write

$$C_\lambda = \sqrt{\frac{\pi e^2}{\omega}} (Q_\lambda + iP_\lambda),$$  \hspace{1cm} (34.33)

which serves to define $Q_\lambda$ and $P_\lambda$. Then Eq. (34.30) becomes

$$H = \frac{1}{2} \sum_\lambda \omega_\lambda (Q_\lambda^2 + P_\lambda^2).$$  \hspace{1cm} (34.34)

Here we have written $\omega_\lambda$ instead of $\omega$ to emphasize that $\omega$ depends on $k$ (so $\omega$ cannot be taken out of the sum).

Since there have been several guesses in the derivation of this Hamiltonian, we should now check to see that it really does give the correct equations of motion. But this is relatively easy. We have

$$\dot{Q}_\lambda = \frac{\partial H}{\partial P_\lambda} = \omega P_\lambda, \hspace{1cm} \dot{P}_\lambda = -\frac{\partial H}{\partial Q_\lambda} = -\omega Q_\lambda,$$  \hspace{1cm} (34.35)

which imply Eq. (34.21).

Altogether, we have shown that the free electromagnetic field possesses one degree of freedom for each mode, and that the evolution of each mode is that of a harmonic oscillator.

We turn now to the case of the electromagnetic field interacting with matter. At first we suppose that some charge density $\rho(r,t)$ and current density $J(r,t)$ are simply given as functions of space and time. Of course, these densities must satisfy the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0,$$  \hspace{1cm} (34.36)

but apart from that they are arbitrary. Now Maxwell’s equations are

$$\nabla \cdot E = 4\pi \rho, \hspace{1cm} \nabla \cdot B = 0,$$

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial E}{\partial t}, \hspace{1cm} \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t},$$  \hspace{1cm} (34.37)

and the electric and magnetic fields are given in terms of the potentials by

$$E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t},$$  \hspace{1cm} (34.38a)

$$B = \nabla \times A.$$  \hspace{1cm} (34.38b)

Equations (34.38) cause the homogeneous Maxwell equations to be satisfied identically, and they cause the inhomogeneous Maxwell equations to become

$$\nabla^2 \phi = -4\pi \rho - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot A),$$  \hspace{1cm} (34.39)
\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J} + \nabla \left( \frac{1}{c} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} \right). \] (34.40)

This is in an arbitrary gauge. When we impose Coulomb gauge, there is some simplification. In particular, Eq. (34.39) becomes the Poisson equation,

\[ \nabla^2 \phi = -4\pi \rho, \] (34.41)

which has the solution

\[ \phi(r, t) = \int d^3r' \frac{\rho(r', t)}{|r - r'|}. \] (34.42)

There are two remarks to make about this result, which is familiar from electrostatics. The first concerns the retardation which must be present in all physically measurable fields. As we see from Eq. (34.42), in Coulomb gauge the scalar potential at one point of space is determined by the charge density at all other points of space evaluated at the same time. There is no retardation. This is not a violation of causality, since \( \phi \) is not directly measurable.

What is directly measurable is the total electric field. In Coulomb gauge, the longitudinal and transverse parts of the electric field are precisely the two terms seen in Eq. (34.38a),

\[ \mathbf{E}_\parallel = -\nabla \phi, \quad \mathbf{E}_\perp = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \] (34.43)

so we see that \( \mathbf{E}_\parallel \), like \( \phi \), has no retardation. However, it turns out that causality is saved by \( \mathbf{E}_\perp \), which has an instantaneous (non-retarded) part which exactly cancels the non-retarded part of \( \mathbf{E}_\parallel \), to give a total electric field which is properly retarded.

The second remark to make about Eq. (34.42) is that it shows that \( \phi \) is not a dynamical variable, for if \( \rho \) is given as a function of \((r, t)\), as we assume, then by an integration \( \phi \) is determined as a function of \((r, t)\). There is no equation of evolution, and there are no initial conditions. To state this another way, if \( \rho \) is regarded as a function of the matter degrees of freedom, as we shall do momentarily, then \( \phi \) also is a function of the matter degrees of freedom. It is not an independent dynamical variable.

Let us now switch over to k-space in the various field equations we have written down, using periodic boundary conditions in a box of volume \( V \). We begin with the continuity equation (34.36), which becomes

\[ \dot{\rho}_k + i \mathbf{k} \cdot \mathbf{J}_k = 0. \] (34.44)

This equation gives the longitudinal component of the current in terms of \( \rho \). We write it in a slightly different form:

\[ \mathbf{J}_{\parallel k} = \frac{1}{k^2} \mathbf{k}(\mathbf{k} \cdot \mathbf{J}_k) = \frac{i}{k^2} \dot{\rho}_k. \] (34.45)
Next, the Poisson equation, Eq. (34.41), becomes
\[ k^2 \phi_k = 4 \pi \rho_k, \]
which may be combined with Eq. (34.45) to give
\[ J_{||k} = \frac{i k}{4 \pi} \phi_k. \]
Finally, setting \( \nabla \cdot A = 0 \) and switching to \( k \)-space, Eq. (34.40) becomes
\[ k^2 A_k + \frac{1}{c^2} \dot{A}_k = \frac{4 \pi}{c} J - \frac{i k}{c} \phi_k. \]
Since the left hand side is purely transverse and since the second term on the right hand side is purely longitudinal, this second term must cancel the longitudinal part of the first term on the right. Indeed, we see by Eq. (34.47) that this is so, and that the transverse components of the vector potential are driven by the transverse components of the current. Multiplying the result through by \( c^2 \), we obtain
\[ \ddot{A}_k + \omega^2 A_k = 4 \pi c J_{\perp k}. \]
This replaces Eq. (34.9) for the interacting field; the equation of motion for \( A_k \) is now a driven harmonic oscillator.

Just as in the case of the free field, it is convenient to express the vector potential as a sum over the modes of the field. Therefore we take over the Fourier series (34.5) to the case of the interacting field, which defines the Fourier coefficients \( A_k \) and their time derivatives \( \dot{A}_k \). We do not yet have a definition of the mode amplitudes \( C_k \) or \( C \) for the interacting field, but we can obtain such a definition by calling on the results for the free field. In particular, let us adopt Eq. (34.25) as the definition of the vectors \( C_k \) in terms of \( (A_k, \dot{A}_k) \) for the interacting field, and then define \( C_\lambda = C_{k\mu} \) by \( C_k = \sum_{\mu} C_{k\mu} \epsilon_{k\mu} \). Then Eqs. (34.24) and the Fourier series (34.22) and (34.23) are valid also for the interacting field. What are not valid for the interacting field are the equations of motion (34.21).

We see that the field degrees of freedom are the same in the case of the interacting field as they were for the free field; they are the transverse components of the vector potential. The longitudinal component of the vector potential vanishes by the Coulomb gauge condition, and the scalar potential is not a dynamical variable.

We now incorporate the matter degrees of freedom into our system. We assume the matter consists of \( n \) nonrelativistic particles with masses \( m_\alpha \), charges \( q_\alpha \), and positions \( x_\alpha \), \( \alpha = 1, \ldots, n \). We use the symbol \( x \) to denote the positions of the particles, and reserve the
symbol \( r \) for a generic field point. Then the charge and current densities are given by

\[
\rho(r, t) = \sum_{\alpha} q_{\alpha} \delta(r - x_{\alpha}(t)) \tag{34.50a}
\]

\[
\mathbf{J}(r, t) = \sum_{\alpha} q_{\alpha} \mathbf{x}_{\alpha}(r - x_{\alpha}(t)), \tag{34.50b}
\]

which satisfy the continuity equation. The particles obey the equations of motion,

\[
m_{\alpha} \ddot{x}_{\alpha} = q_{\alpha} \left[ \mathbf{E}(x_{\alpha}) + \frac{1}{c} \mathbf{x}_{\alpha} \times \mathbf{B}(x_{\alpha}) \right]. \tag{34.51}
\]

Let us now look for a Hamiltonian for the combined matter-field system. As before, we guess that it is the total energy, which is the kinetic energy of the matter plus the field energy. We call this energy \( H \), and write

\[
H = \sum_{\alpha} \frac{m_{\alpha}}{2} |\mathbf{x}_{\alpha}|^2 + \frac{1}{8\pi} \int d^3r \left( E^2 + B^2 \right). \tag{34.52}
\]

One can check by direct appeal to the equations of motion (34.51) and Maxwell’s equations that the Hamiltonian (so defined) is conserved. As with the free field, we must now identify \( q \)'s and \( p \)'s, and check that Hamilton’s equations are the same as the known equations of motion.

We begin with the electric field contribution to the field energy. We break the electric field up into its longitudinal and transverse parts, so that

\[
E^2\text{-term} = \frac{1}{8\pi} \int d^3r (E_\parallel + E_\perp)^2. \tag{34.53}
\]

The cross terms in this integral vanish because of the Parseval identity (32.50). As for the longitudinal contribution, it can be transformed,

\[
\frac{1}{8\pi} \int d^3r E_\parallel^2 = \frac{1}{8\pi} \int d^3r |\nabla \phi|^2 = -\frac{1}{8\pi} \int d^3r \phi \nabla^2 \phi = \frac{1}{2} \int d^3r \phi(r, t) \rho(r, t)
\]

\[
= \frac{1}{2} \int d^3r d^3r' \frac{\rho(r, t) \rho(r', t)}{|r - r'|} = \frac{1}{2} \sum_{\alpha\beta} \frac{q_{\alpha} q_{\beta}}{|x_{\alpha} - x_{\beta}|}, \tag{34.54}
\]

where in the second equality we integrate by parts and discard boundary terms, in the third equality we use the Poisson equation (34.41), in the fourth equality we use the solution (34.42) to the Poisson equation, and where in the last equality we use Eq. (34.50a). We see that the longitudinal electric field energy is otherwise just the usual Coulomb energy of interaction among the particles, without retardation. This is nice, but unfortunately the final expression in Eq. (34.54) contains in its diagonal terms \( (\alpha = \beta) \) the infinite self-energies.
of the electrostatic field of the particles. To obtain a finite result and one that is hopefully meaningful, we simply discard these terms, effectively making the replacement,

\[ \frac{1}{8\pi} \int d^3 \mathbf{r} E_\parallel^2 \rightarrow \sum_{\alpha<\beta} \frac{q_\alpha q_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|}. \]  

(34.55)

This takes care of the longitudinal contribution to the field energy.

The rest of the field energy is

\[ \mathcal{H}_{\text{em}} = \frac{1}{8\pi} \int d^3 \mathbf{r} (E_\perp^2 + B^2), \]  

(34.56)

which we identify as the field Hamiltonian. We can express this in terms of the mode amplitudes \( C_\lambda \), using the Fourier series (34.22) and (34.23) and using Eq. (34.43) for \( \mathbf{E}_\perp \).

The calculation is exactly the same as in the free field case, so we obtain the same result,

\[ \mathcal{H}_{\text{em}} = \frac{1}{2\pi} \sum_\lambda \frac{\omega_\lambda^2}{c^2} |C_\lambda|^2 = \sum_\lambda \frac{\omega_\lambda^2}{2}(Q_\lambda^2 + P_\lambda^2), \]  

(34.57)

where in the final form we have taken over the definition (34.33) for \( Q_\lambda \) and \( P_\lambda \) from the case of the free field.

As for the rest of the Hamiltonian (the matter kinetic energy plus the longitudinal field energy), we identify this as the matter Hamiltonian. We guess that the usual definition of the canonical momentum \( \mathbf{p}_\alpha \) in a nonrelativistic mechanical systems is appropriate here,

\[ \mathbf{p}_\alpha = m_\alpha \dot{\mathbf{x}}_\alpha + \frac{q_\alpha}{c} \mathbf{A} (\mathbf{x}_\alpha), \]  

(34.58)

and we express the matter Hamiltonian as a function of \( \mathbf{x}_\alpha \) and \( \mathbf{p}_\alpha \). This gives

\[ \mathcal{H}_{\text{matter}} = \sum_\alpha \frac{1}{2m_\alpha} \left( \mathbf{p}_\alpha - \frac{q_\alpha}{c} \mathbf{A} (\mathbf{x}_\alpha) \right)^2 + \sum_{\alpha<\beta} \frac{q_\alpha q_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|}. \]  

(34.59)

The matter Hamiltonian also depends on the field \( q \)'s and \( p \)'s, since \( \mathbf{A} \) depends on the \( C_\lambda \) according to Eq. (34.22), and the \( C_\lambda \) depend on the \( Q_\lambda \) and \( P_\lambda \) through Eq. (34.33). The total Hamiltonian is

\[ \mathcal{H} = \mathcal{H}_{\text{matter}} + \mathcal{H}_{\text{em}}. \]  

(34.60)

Again, we have made several guesses and must check that this Hamiltonian gives the correct equations of motion, according to Hamilton’s equations. These equations of motion can be taken to be Eq. (34.49) (which are equivalent to Maxwell’s equations), and Eq. (34.51). This check will be left as an exercise.
Finally, we make one further cosmetic change in preparation for quantization. We define new variables $a_\lambda$,

$$a_\lambda = \frac{Q_\lambda + iP_\lambda}{\sqrt{2\hbar}},$$

$$a_\lambda^* = \frac{Q_\lambda - iP_\lambda}{\sqrt{2\hbar}},$$

which are obviously just classical analogs of annihilation and creation operators in quantum mechanics. We call these variables *normal variables*. The normal variables are related to the $C_\lambda$ by

$$C_\lambda = \sqrt{\frac{2\pi\hbar c^2}{\omega_\lambda}} a_\lambda.$$  

(34.62)

Expressing the vector potential and field Hamiltonian in terms of the normal variables, we have

$$A(r) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_\lambda \frac{1}{\sqrt{\omega_\lambda}} \left[ a_\lambda \epsilon_\lambda e^{ikr} + \text{c.c.} \right],$$

(34.63)

and

$$H_{em} = \sum_\lambda \hbar\omega_\lambda |a_\lambda|^2.$$  

(34.64)