Now we use some of the things learned in the analysis of natural line width to study other problems. We begin with the divergences in the K-H formula. For simplicity consider elastic scattering, where $A+\gamma \to A+\gamma'$, $A=$ gnd state. The cross section is

$$\frac{d\sigma}{d\Omega'} = r_e^2 \, |M|^2$$

where

$$M = \hat{\varepsilon}\cdot\hat{\varepsilon}'^* + \sum_{I\neq A}\left[\frac{\langle A|\vec{\beta}\cdot\hat{\varepsilon}'^*|I\rangle\langle I|\vec{\beta}\cdot\hat{\varepsilon}|A\rangle}{\omega-\omega_{IA}} - \frac{\langle A|\vec{\beta}\cdot\hat{\varepsilon}|I\rangle\langle I|\vec{\beta}\cdot\hat{\varepsilon}'^*|A\rangle}{\omega+\omega_{IA}}\right]$$

(see p. 9, 3/20/08). This term ↑ is resonant when $\omega$ (freq. of incident photon) $= \omega_{IA}$ (resonant freq. connecting gnd state A with some excited state I).

The resonant denom is

$$\omega-\omega_{IA} = \omega - E_I + E_A \qquad (\hbar=1 \text{ here}).$$

It comes out of the Dyson series by doing a time integral over

$$e^{i(\omega-E_I+E_A)t} = e^{-i(E_I-E_A-\omega)t}.$$

The phase factor $e^{-iE_It}$ is the time evolution of the state I computed by the unpert. Ham. $H_0$. The Dyson series is one of successive approximations, in which we insert one approximation for the t-evolution (in this case, the unpert. evol.) into a t-integral to get the next approximation.

However, we know that the actual time evolution of the intermed. state I is more closely given by

$$e^{-i(E_I+\Delta E_I^{(1)}+\Delta E_I^{(2)}-i\Gamma/2)t} = \text{(cont'd)}$$

$$= e^{-i\tilde{E}_I t - \Gamma t/2},$$

where

$$\tilde{E}_I = E_I + \Delta E_I^{(1)} + \Delta E_I^{(2)}$$

is the real part of the energy, corrected to 2nd order by interactions with the field. See for the energy corrections, in a general notation. Thus on using this more accurate t-evolution for the state I in the t-integral, we get a modified denominator in the K-H formula:

$$\omega - \tilde{E}_I + E_A + i\Gamma/2 .$$

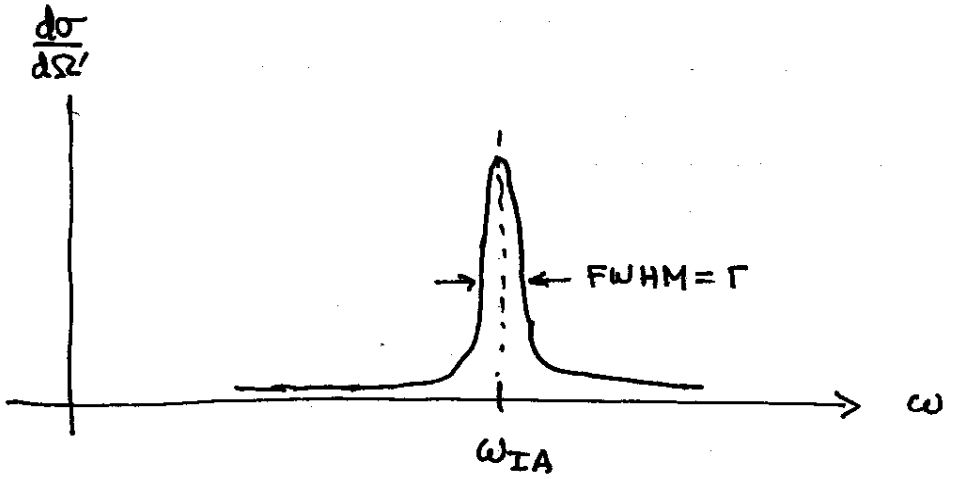
The energy of state A $(E_A)$ should also be modified by $\Delta E_A^{(1)}$ and $\Delta E_A^{(2)}$, but being the ground state it has an $\infty$ life time and so $\Gamma_A = 0$. Thus we should really use $\omega - \tilde{E}_I + \tilde{E}_A + i\Gamma/2$, where $\Gamma$ refers to state I. Let's henceforth interpret $\omega_{IA}$ as $\tilde{E}_I - \tilde{E}_A$. Then the K-H denominator is $\omega - \omega_{IA} + i\Gamma/2$, and it no longer vanishes when $\omega = \omega_{IA}$.

When $\omega \approx \omega_{IA}$, one term dominates in the K-H formula,

$$M \approx \frac{\langle A|\vec{p}\cdot\hat{\varepsilon}'^*|I\rangle\langle I|\vec{p}\cdot\hat{\varepsilon}|A\rangle}{\omega - \omega_{IA} + i\Gamma/2},$$

$$\frac{d\sigma}{d\Omega'} = \alpha^4 a_0^2 \frac{|\langle A|\vec{p}\cdot\hat{\varepsilon}'^*|I\rangle|^2 \, |\langle I|\vec{p}\cdot\hat{\varepsilon}|A\rangle|^2}{(\omega - \omega_{IA})^2 + \Gamma^2/4} .$$

To sketch the cross-section as a fn. of incident photon $\omega$ we have something like (not to scale):




Near $\omega = \omega_{IA}$ the one term expression (prev. page) gives the same Lorentzian dependence on $\omega$ as we had for the natural line width. The FWHM is $\Gamma/\hbar$ (in freq. units), which for E1 transitions at optical frequencies is $O(\alpha^3)$ in a.u. So if the freq. range in the plot above is $O(1)$ in a.u. (i.e. typical optical freq's) then the width of the resonance is $\sim 10^{-6}$. For $\omega$ not near a resonance the K-H denominators are all $O(1)$ in a.u., so $d\sigma/d\Omega' \sim \alpha^4 a_0^2$, pretty small by atomic standards. But on resonance ($\omega = \omega_{IA}$) the denominator is $\sim \Gamma^2 \sim \alpha^6 \sim 10^{-13}$, so the height of the peak is $\sim 10^{13}$ × higher than the nonresonant values. This becomes over $10^{20}$ for M1 transitions. The total area under the resonant peak (width $\sim \Gamma$, height $\sim 1/\Gamma^2$) is order $1/\Gamma$, so in broad band radiation by far most of the scattering is

resonant, at least until the beam is depleted in the resonant frequency range. This is the explanation for the dark Fraunhofer lines in the solar spectrum (resonant scattering by cooler gases in the sun's atmosphere of the radiation coming from the photosphere).

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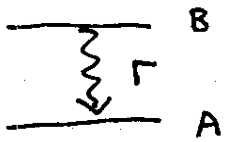
Next we examine the real parts of the energy shift $\Delta E_i^{(1)}$ and $\Delta E_i^{(2)}$ due to the interaction of atomic electrons with the EM field. In general notation we have

$$\Delta E_i^{(1)} = \langle i | V | i \rangle ,$$

$$\text{and} \quad \Delta E_i^{(2)} = \sum_{k \neq i} \left( \frac{P}{E_i - E_k} \right) |\langle k | V | i \rangle|^2 .$$

We are not talking about the interaction of atomic electrons with real photons — that would be photon scattering, described by the K-N formula. Instead, we take the case that the photon state is the vacuum. Nevertheless, even in the vacuum there are fluctuations of the EM field, and the effects we shall find can be ascribed to the interaction of the electrons with those fluctuations.

The imaginary part $\Gamma$ of the 2nd order energy shift is the decay rate ($w = \Gamma/\hbar$) of an excited state B, but what about the real energy shifts? To calculate these we must switch from the general notation above to notation specific



to the atom.

For the initial state $|i\rangle$, write

$$|i\rangle \longrightarrow |B\rangle|0\rangle \equiv |B0\rangle$$

$$E_i \longrightarrow E_B.$$

For the perturbing Hamiltonian, write

$$V \longrightarrow H_1 + H_2$$

where

$$H_1 = \frac{e}{mc}\vec{p}\cdot\vec{A}(\vec{x})$$

$$H_2 = \frac{e^2}{2mc^2}\vec{A}(\vec{x})^2$$

It's better to use ordinary units in this calculation. Then at 1st order we have

$$\Delta E_B^{(1)} = \langle B0|H_1|B0\rangle.$$

But since $H_1 \sim \vec{A}(\vec{x})$ can only create or destroy one photon, and since it is sandwiched between states with equal numbers of photons (namely, 0), the result is 0. We must go to 2nd order to get an energy shift.

At 2nd order we have a sum on intermediate states $k$ (in general notation), which in principle is a sum over a complete set of states for the matter and field. But since $|i\rangle \to |B0\rangle$ has no photons and since $H_1$ can only create or destroy one photon, the only intermediate states $k$ that are reachable from $|B0\rangle$ are those with precisely one

photon. So we identify

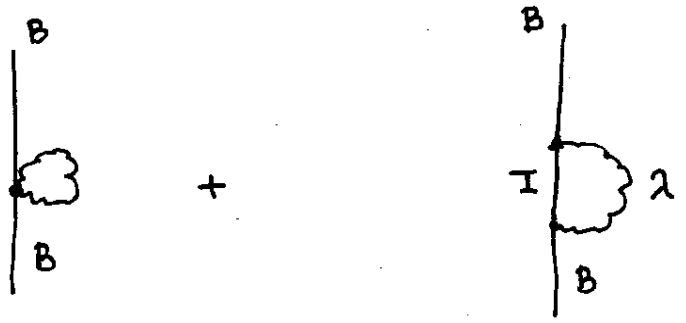
$$|k\rangle \longrightarrow |I\rangle a^{+}_{\lambda}|0\rangle \equiv |I\lambda\rangle$$

$$E_k \longrightarrow E_I + \hbar\omega$$

where $I$ is an intermediate atomic state and $\omega = \omega_\lambda$. Then the energy shift at 2nd order and associated diagrams are

$$\Delta E_B^{(2)} = \langle B0|H_2|B0\rangle + \sum_{I\lambda} \frac{\langle B0|H_1|I\lambda\rangle\langle I\lambda|H_1|B0\rangle}{E_B - E_I - \hbar\omega},$$

where the principal value is understood in the 2nd term.




The diagrams are the same as in the scattering of a photon (K-H formula), except the external photon lines are joined to form an internal loop.

The first diagram (from $H_2$) does not contribute to the effect we wish to calculate, so we will ignore it. The reason will be left as an exercise. The second diagram can be made more explicit by substituting $H_1 = \frac{e}{mc}\vec{p}\cdot\vec{A}(\vec{x})$, then using the Fourier series for $\vec{A}(\vec{x})$,

$$\vec{A}(\vec{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_\lambda \frac{1}{\sqrt{\omega}} \left[ a_\lambda \hat{\varepsilon}_\lambda e^{i\vec{k}\cdot\vec{r}} + a_\lambda^+ \hat{\varepsilon}_\lambda^* e^{-i\vec{k}\cdot\vec{r}} \right],$$

and then carrying out the field part of the matrix element. This gives

$$\Delta E_B^{(2)} = \left(\frac{e}{mc}\right)^2 \frac{2\pi\hbar c^2}{V} \sum_I \sum_\mu \sum_{\vec{k}} \frac{1}{\omega} \frac{|\langle I| \vec{p}\cdot\hat{\varepsilon}_\lambda^* e^{-i\vec{k}\cdot\vec{x}} |B\rangle|^2}{E_B - E_I - \hbar\omega}.$$

First we make the electric dipole approximation, $e^{-i\vec{k}\cdot\vec{x}} \approx 1$, which is valid for optical photons but not for photons with energy $\hbar\omega$ too much above $\alpha mc^2$. (Optical photons have energies $\sim \alpha^2 mc^2$). This implies a cut off at high photon energies, which we will return to later. We are using nonrelativistic QM, so our calculations are only valid for NR energies anyway.

With this approximation, the only part of the integrand that depends on the polarization $\mu$ of the photon or the direction of $\vec{k}$ is $\hat{\varepsilon}_\lambda$.

Define

$$\vec{p}_{IB} = \langle I|\vec{p}|B\rangle,$$

take $V \to \infty$ so

$$\sum_{\vec{k}} \to \frac{V}{(2\pi)^3} \int_0^\infty k^2 dk \int d\Omega_k .$$

Then

$$\sum_\mu \int d\Omega_k \; |\hat{\varepsilon}_\lambda^* \cdot \vec{p}_{BI}|^2 = \frac{8\pi}{3} |\vec{p}_{BI}|^2,$$

and we have

$$\Delta E_B^{(2)} = \frac{2}{3\pi}\frac{e^2}{m^2}\hbar \sum_I |\vec{p}_{BI}|^2 \int_0^\infty k^2 dk \frac{1}{\omega} \frac{1}{E_B - E_I - \hbar\omega}.$$

Change variable of integration to $E = \hbar\omega$ ($k = E/\hbar c$), and we get

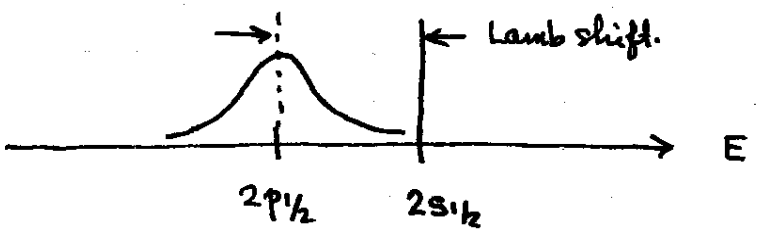
$$\boxed{\Delta E_B^{(2)} = \frac{4}{3\pi}\left(\frac{e^2}{\hbar c}\right)\frac{1}{mc^2}\frac{1}{2m}\sum_I |\vec{p}_{BI}|^2 \int_0^\infty \frac{E\,dE}{E_B - E_I - E}.}$$

The integral diverges because of the upper limit $\infty$. It is a kind of ultraviolet catastrophe. Since we are using NR QM we might guess that relativistic effects would impose a cutoff on the upper limit of the integral, say $E_{max} \sim mc^2$, but then the integral diverges linearly in the cutoff and in the absence of any better guess for the cutoff we do not have a useful result.

This divergent integral is an example of the kind of infinities that plagued QED in its first 20 years of existence. They occur whenever QED is pushed beyond lowest order. The lowest order results agree very well with experiment (e.g., the life time of excited states), but cannot be corrected to higher order (with this primitive version of the theory).

Prior to WW II no one knew how to interpret the energy shift $\Delta E_B^{(2)}$ above, but on the other hand there was no clear disagreement between

theory and experiment regarding atomic energy levels. All experimental results seemed adequately explained by the Dirac equation, without need for ~~the~~ any extra shift $\Delta E_B^{(2)}$. That situation changed in 1947 with the announcement by Lamb and Retherford of the splitting between the $2s_{1/2}$ and $2p_{1/2}$ levels of hydrogen, measured rather accurately by microwave techiques at about 1.05 GHz. The 2p state makes an E1 transition to the 1s state in $\sim 10^{-9}$ sec, so it has a natural line width comparable to the Lamb shift, and the energy level diagram looks something like this:



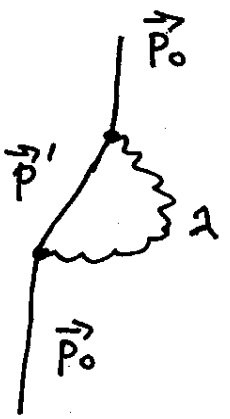

The $2s_{1/2}$ state also has a natural line width, but it is very small on the scale of the diagram.

The suspicion immediately arose that the $\Delta E_B^{(2)}$ above, properly interpreted, would explain the Lamb shift. This was quickly done by Bethe, who used some of the renormalization ideas that had been under development by Kramers for some time. Bethe had the intuition that the phenomenon was essentially nonrelativistic, and so is his analysis. We now explain Bethe's calculation.

First we consider the second order energy shift for a free electron (not bound in an atom) due to interactions with the EM field. This is a little easier than what we did above, since there is no atom to complicate the QM. The unpert. Ham is now

$$H_0 = \frac{p^2}{2m} + \sum_\lambda \hbar\omega_\lambda a_\lambda^+ a_\lambda$$

with $H_1, H_2$ as before. The unpert. eigenstates of the electron are normalized plane waves $|\vec{p}\rangle$ in a box (later we let $V \to \infty$). We assume the initial state of the electron is $|\vec{p}_0\rangle$. The Feynman diagram for the $H_1^2$ process is




"self-energy diagram"

where $\vec{p}\,'$ is the momentum of the intermediate state. The energy shift is

$$\Delta E^{(2)}_{free} = \sum_{\vec{p}\,'} \sum_\lambda \frac{\langle \vec{p}_0 0 | H_1 | \vec{p}\,' \lambda\rangle \langle \vec{p}\,' \lambda | H_1 | \vec{p}_0 0\rangle}{\frac{p_0^2}{2m} - \frac{p'^2}{2m} - \hbar\omega}.$$

Doing the field matrix elements this becomes

$$\Delta E^{(2)}_{\text{free}} = \left(\frac{e}{mc}\right)^2 \left(\frac{2\pi\hbar c^2}{V}\right) \sum_{\vec{p}'} \sum_{\mu} \sum_{\vec{k}} \frac{1}{\omega} \frac{\left|\langle \vec{p}' | \vec{p}\cdot\hat{\varepsilon}^*_\lambda \, e^{-i\vec{k}\cdot\vec{x}} | \vec{p}_0\rangle\right|^2}{\frac{p_0^2}{2m} - \frac{p'^2}{2m} - \hbar\omega}$$

The $\vec{p}$ inside the matrix element is the operator $-i\hbar\nabla$, so acting on $e^{-i\vec{k}\cdot\vec{x}}$ and the plane wave $|\vec{p}_0\rangle$, it gets replaced by the c-number

$$\vec{p} \to \vec{p}_0 - \hbar\vec{k}.$$

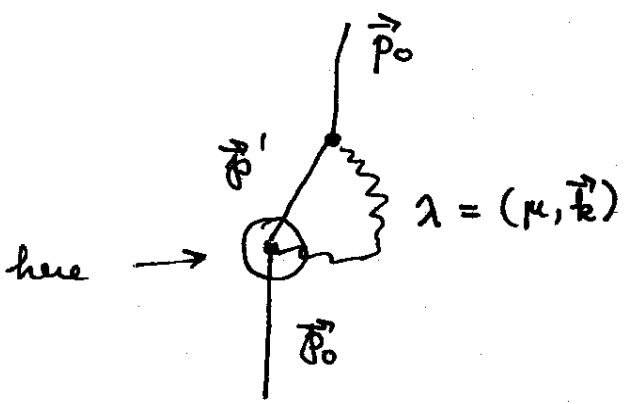
However, $\vec{k}\cdot\hat{\varepsilon}_\lambda = 0$, so the numerator becomes

$$\text{num.} = \left| \hat{\varepsilon}^*_\lambda \cdot \vec{p}_0 \, \langle \vec{p}' | e^{-i\vec{k}\cdot\vec{x}} | \vec{p}_0\rangle \right|^2$$

Now the remaining matrix element is a Kronecker $\delta$,

$$\langle \vec{p}' | e^{-i\vec{k}\cdot\vec{x}} | \vec{p}_0\rangle = \delta_{\vec{p}', \vec{p}_0 - \hbar\vec{k}},$$

which represents momentum conservation at the vertex of the Feynman diagram:



Thus the $\vec{p}'$ sum can be done, and we have

$$\Delta E^{(2)}_{free} = \frac{e^2}{m^2}\frac{2\pi\hbar}{V}\sum_{\mu}\sum_{\vec{k}}\frac{1}{\omega}\frac{|\hat{\varepsilon}^*_\lambda\cdot\vec{p}_0|^2}{\frac{p_0^2}{2m}-\frac{(\vec{p}_0-\hbar\vec{k})^2}{2m}-\hbar\omega}.$$

Now look at the denominator:

$$\frac{p_0^2}{2m}-\frac{(\vec{p}_0-\hbar\vec{k})^2}{2m}-\hbar\omega = \frac{p_0^2}{2m}-\frac{p_0^2}{2m}+\frac{\hbar\,\vec{p}_0\cdot\vec{k}}{m}-\frac{\hbar^2k^2}{2m}-\hbar\omega.$$

The first two terms cancel and if $\frac{p_0^2}{2m}, \hbar\omega \ll mc^2$, then the next two terms are negligable, too. Then the denominator becomes simply $-\hbar\omega$. (A basic rule: at NR energies, photons contribute energy, but not much momentum.)

Now the only place the summand depends on $\mu$ or the direction of $\vec{k}$ is in $\hat{\varepsilon}_\lambda$. Take $V\to\infty$, so

$$\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int_0^\infty k^2dk\int d\Omega_k,$$

and use

$$\sum_{\mu}\int d\Omega_k\,|\hat{\varepsilon}^*_\lambda\cdot\vec{p}_0|^2 = \frac{8\pi}{3}p_0^2.$$

Then

$$\Delta E^{(2)}_{free} = \frac{42}{3\pi}\frac{e^2}{m^2}\hbar\int_0^\infty k^2dk\,\frac{1}{\omega}\,\frac{p_0^2}{(-\hbar\omega)}$$

or with $E = \hbar\omega = \hbar c k$,

$$\boxed{\Delta E^{(2)}_{free} = -\frac{4}{3\pi}\left(\frac{e^2}{\hbar c}\right)\frac{1}{mc^2}\,\frac{p_0^2}{2m}\int_0^\infty dE} \qquad (E = \text{photon energy})$$

This integral also diverges linearly, and has the same problems as the integral for $\Delta E_B^{(2)}$ if we impose a cutoff on the photon energy $E_{max}$.

Let's put an unknown cutoff $E_{max}$ on the upper limit to this integral and write

$$C = -\frac{4}{3\pi}\frac{\alpha}{mc^2}\int_0^{E_{max}} dE,$$

so

$$\Delta E^{(2)}_{free} = C\,\frac{p_0^2}{2m}.$$

Notice that if $E_{max} \sim mc^2$, then $C \sim \alpha$ = small, but if we take $E_{max} \to \infty$, then $C \to \infty$. ($C$ is dimensionless.) Then the total energy of the free electron, through 2nd order pert'n theory, is

$$E^{tot}_{free} = \frac{p_0^2}{2m} + C\,\frac{p_0^2}{2m} = (1+C)\,\frac{p_0^2}{2m}.$$

We cannot switch off the interaction of the electron with the EM field, so when we measure the energy of an electron with momentum

$\vec{p}_0$ what we get is $(1+C)\frac{p_0^2}{2m}$, not $\frac{p_0^2}{2m}$. Thus we must interpret

$$(1+C)\frac{p_0^2}{2m} \quad \text{as} \quad \frac{p_0^2}{2m_{obs}},$$

where $m_{obs}$ is the observed mass of the electron, which differs from the mass parameter $m$ that enters into the Hamiltonian:

$$\frac{1+C}{m} = \frac{1}{m_{obs}},$$

$$m_{obs} = m(1-C) \quad \text{if } C \text{ small.} \quad (\text{note } C<0, \text{ so } m_{obs}>m).$$

The mass parameter $m$ is the <u>bare mass</u>, which is not observable.

If we model a classical electron as a particle with a mechanical mass $m$, then the observed mass must be more because of the energy (= mass by relativity) that is contained in the electrostatic field. Since the electric field (magnitude) is $E = e/r^2$, the total mass contained ~~in~~ at radii $r>R$ is

$$\frac{1}{c^2}\int_{r>R} d^3\vec{r}\,\frac{e^2}{8\pi r^4} = \frac{1}{c^2}\int_{r>R} d^3\vec{r}\,\frac{E^2}{8\pi} = \frac{4\pi}{c^2}\int_R^\infty r^2 dr\,\frac{e^2}{8\pi r^4}$$

$$= \frac{1}{2}\frac{e^2}{Rc^2}.$$

If we set this $= m$ and solve for $R$, we get $R = \frac{1}{2} r_e$, $r_e = \frac{e^2}{mc^2}$, the "classical radius of the electron". If we take $R \to 0$, we get $\infty$ (the $\infty$ self-mass of the electron).

Suppose we use photons of maximum energy $E_{max}$ to probe the field of the electron. ~~this takes as~~ These have minimum wavelength of order $\frac{\hbar c}{E_{max}}$. The total mass contained in the field outside this distance is

$$\sim \frac{e^2}{\hbar c}\frac{E_{max}}{c^2} = \alpha \frac{E_{max}}{c^2}.$$

But this is the same order of magnitude as $Gm$, so the correction between $m$ and $m_{obs}$ can be regarded as being due to the electron self energy. If we take the limit $E_{max} \to \infty$, we get the full (infinite) self energy of the electron.

The infinite self energy of the electron is an inconsistency in classical EM theory that cannot be cured at the classical level. It can only be avoided by not looking at problems too carefully (e.g., in books on classical E+M). The difficulty can only be cured at the quantum level. Kramers

might have made greater contributions to renormalization theory, but he believed the correct strategy was to cure the classical infinities first, then quantize.

Return to the Lamb shift. The Lamb shift was discovered experimentally as a splitting between two levels that were predicted on the basis of conventional theory (the Dirac equ) to be degenerate, but for the purposes of the following discussion let's define the Lamb shift for a single level as the difference between the Dirac prediction and the experimental value. ~~If our the~~ By this definition, the Lamb shift is not the energy $\Delta E_B^{(2)}$ ~~[illegible]~~ Here is why.

If our theory is right, then the true (experimental) energy level of an atom must be

$$E_B^{tot} = E_B + \Delta E_B^{(2)}$$

$$= \langle B | \frac{p^2}{2m} + V(\vec{x}) | B \rangle + \Delta E_B^{(2)}$$

The first term, however, is not the predictions of conventional theory; rather, that is

$$E_B^{conv} = \langle B | \frac{p^2}{2m_{obs}} + V(\vec{x}) | B \rangle .$$

When we calculate energy levels we get a formula that contains the mass of the electron, and we must insert the observed mass to get the conventional prediction of the energy level. We could not use the bare mass if we wanted to, because we have no value for it. But in the expression for $E_B^{tot}$ above, it is the bare mass that appears, because we are working out the predictions of quantum mechanics,

starting from the Hamiltonian which is parameterized by the bare mass. Thus

$$\Delta E_B^{lamb} = E_B^{tot} - E_B^{conv}$$

$$= \langle B | \frac{p^2}{2m} - \frac{p^2}{2m_{obs}} | B \rangle + \Delta E_B^{(2)}$$

$$= -C \langle B | \frac{p^2}{2m} | B \rangle + \Delta E_B^{(2)},$$

where we use

$$\frac{1}{m}(1+C) = \frac{1}{m_{obs}}.$$

Both terms in $\Delta E_B^{lamb}$ involve divergent integrals (if $E_{max} \to \infty$); the actual value of the shift is the difference betw. the divergent terms.

To make the first term look like the second, insert a resolution of the identity so

$$\langle B | \frac{p^2}{2m} | B \rangle = \frac{1}{2m} \sum_I \langle B | \vec{p} | I \rangle \cdot \langle I | \vec{p} | B \rangle = \frac{1}{2m} \sum_I |\vec{p}_{BI}|^2.$$

Then using C (~~[illegible]~~), we get

$$\Delta E_B^{lamb} = \frac{4}{3\pi} \frac{\alpha}{mc^2} \frac{1}{2m} \sum_I |\vec{p}_{BI}|^2 \underbrace{\int_0^{E_{max}} dE \left(1 + \frac{E}{E_B - E_I - E}\right)}$$

$$\hookrightarrow (E_B - E_I) \int_0^{E_{max}} \frac{dE}{E_B - E_I - E}.$$

The integral is still divergent, but now only logarithmically so, so

its value is rather insensitive to the cutoff used.

The integral must be done using the principal value prescription. The result is

$$\Delta E_B^{Lamb} = \frac{4}{3\pi}\frac{\alpha}{mc^2}\sum_I \frac{|\vec{p}_{BI}|^2}{2m}\,(E_I - E_B)\ln\left(\frac{E_{max}}{|E_I - E_B|}\right).$$

To further boil this down, we need to do the sum on I. This is purely a problem in atomic physics. Standard tricks are used, which are summarized in Sakurai's book. At the end some numerical work is required. The result vanishes except for $\ell = 0$ (s-waves); in that case, for hydrogen,

$$\Delta E_{Lamb} = \frac{4}{3\pi}\frac{\alpha^3}{n^3}\ln\left(\frac{E_{max}}{\langle|E_I - E_B|\rangle}\right)\frac{e^2}{a_0},$$

where $n$ is the principal qu. # and the average under the log. is defined in a certain way by Bethe. ($e^2/a_0$ = atomic unit of energy) With the cutoff $E_{max} = mc^2$, Bethe got the value 1.04 GHz, very close to the experimental value. This was the first successful application of renormalization theory.

In a proper relativistic calculation, one finds that both energy shifts (for the free and bound electrons) diverge only logarithmically in the cutoff $E_{max}$, not linearly as here. Thus, the difference between the divergent integrals actually converges, and no cutoff is required.

The proper relativistic calculation ~~was~~ of the Lamb shift was first carried out by French and Weisskopf, later Feynman and Schwinger. See the reprint on the story of the Lamb shift for more on the history.