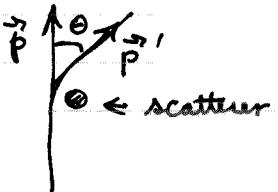


(9)

Now we can do traces.

$$\frac{1}{2} \text{tr} \left[\gamma^0 \left(\frac{\vec{p}+m}{2m} \right) \gamma^0 \left(\frac{\vec{p}'+m}{2m} \right) \right] = \frac{1}{8m^2} \text{tr} \left[\gamma^0 p \gamma^0 p' + m^2 \gamma^0 \gamma^0 \right]$$

$$= \frac{4}{8m^2} \left(\cancel{E} \cancel{E}' - \cancel{(p \cdot p')} + E'E + m^2 g^{00} \right)$$



Use

$$g^{00} = 1$$

$$E = E' \quad (\text{conserv. of energy})$$

$$p \cdot p' = E^2 - \vec{p} \cdot \vec{p}' \quad \text{and note } |\vec{p}| = |\vec{p}'|,$$

$$m^2 = E^2 - |\vec{p}|^2$$

$$\rightarrow = \frac{1}{2m^2} \left(2E^2 - |\vec{p}|^2 + \vec{p} \cdot \vec{p}' \right) = \frac{1}{2m^2} \left(2E^2 - |\vec{p}|^2 (1 - \cos\theta) \right)$$

$$= \frac{1}{m^2} \left(E^2 - |\vec{p}|^2 \sin^2 \theta/2 \right) = \frac{E^2}{m^2} \left(1 - \beta^2 \sin^2 \frac{\theta}{2} \right), \quad \beta = \frac{v}{c} = \frac{|\vec{p}|}{E}.$$

So,

$$\frac{d\sigma}{d\Omega} = 2\pi e^2 E^2 |\tilde{\Phi}(\vec{q})|^2 \left(1 - \beta^2 \sin^2 \frac{\theta}{2} \right).$$

$$\text{For Coulomb case, } \Phi(\vec{x}) = \frac{ze}{|\vec{x}|}, \quad \tilde{\Phi}(\vec{q}) = \frac{4\pi ze}{(2\pi)^{3/2} q^2},$$

get

$$\frac{d\sigma}{d\Omega} = \frac{z^2 e^4 E^2}{4|\vec{p}|^4 \sin^4 \theta/2} \left(1 - \beta^2 \sin^2 \frac{\theta}{2} \right)$$

$\vec{q} = \vec{p}' - \vec{p}$ =
momentum transfer

(Mott cross section).
(relativistic generalization
of Rutherford)

The term $\beta^2 \sin^2 \theta/2$ is of order $(v/c)^2$, obviously a relativistic correction. It is due to the magnetic interactions of the electron

(through its spin) with the potential $\Phi(\vec{x})$. This term is not present in the $d\sigma/d\Omega$ for scattering of spinless bosons.

Now we consider a more sophisticated problem, namely, e^+e^- annihilation. The reaction is



Although the Feynman diagram for single photon annihilation exists ($e^- + e^+ \rightarrow \gamma$)  this process cannot occur in free space because you cannot conserve both energy and momentum in $e^- + e^+ \rightarrow \gamma$. So 2-photon annihilation is the simplest that can occur in free space.

We must now quantize the EM field as well as the e^+e^- field. To do this we return to the ^{level of the} classical Lagrangian, and write

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_{em} + \mathcal{L}_{int},$$

where

$$\mathcal{L}_D = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \quad (\text{free Dirac particle})$$

$$\mathcal{L}_{em} = -\frac{F^{\mu\nu} F_{\mu\nu}}{16\pi} = \frac{E^2 - B^2}{8\pi} \quad (\text{EM field})$$

$$\begin{aligned} \mathcal{L}_{int} &= e \bar{\psi} \gamma^\mu A_\mu \psi = - J_\mu A^\mu \quad (\text{interaction}) \\ &= e \bar{\psi}^+ \Phi \psi - e \bar{\psi}^+ \vec{\alpha} \cdot \vec{A} \psi \end{aligned}$$

where we set $g = -e$. The 3 terms in \mathcal{L} are manifestly Lorentz invariant, as required by an acceptable relativistic theory.

Here $A^\mu = (\Phi, \vec{A})$. In Coulomb gauge, Φ is not an independent dyn. variable, but rather is a fn. of the matter variables, here ψ and $\bar{\psi}$. That is,

$$\Phi(\vec{x}) = \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

where $\rho(\vec{x}) = \bar{\psi} \psi^+(\vec{x}) \psi(\vec{x})$.

This Lagrangian is interpreted here at the "classical" level, which means 1st quantized Dirac field plus unquantized EM field. The eqns of motion are the Dirac eqn + Maxwell's eqns (coupled), i.e.

$$(i\gamma^\mu \partial_\mu - m)\psi = -e A^\mu \psi$$

$$F^{\mu\nu}_{,\nu} = -4\pi J^\mu = \cancel{4\pi e \bar{\psi}} + 4\pi e \bar{\psi} \gamma^\mu \psi. \quad \}$$

We convert \mathcal{L} to a Hamiltonian. Let π be the field conjugate to ψ and $\dot{\vec{\Phi}}$ the field conjugate to \vec{A} . Then

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0$$

$$\dot{\vec{\Phi}} = \frac{\partial \mathcal{L}}{\partial \vec{A}} = \frac{\vec{A}}{4\pi}.$$

The second eqn. follows from fact that

$$\mathcal{L}_{em} = \frac{E^2 - B^2}{8\pi} = \frac{E_{||}^2}{8\pi} + \frac{E_\perp^2 + B^2}{8\pi},$$

where $\vec{E}_{||} = -\nabla \Phi$ and $\vec{E}_\perp = -\dot{\vec{A}}$ (in Coulomb gauge). The momenta conjugate to $\bar{\psi}$ and $\dot{\Phi}$ are zero (there is no $\dot{\bar{\psi}}$ or $\ddot{\Phi}$ in the

Lagrangian). So for the Hamiltonian we get

(12)

$$\mathcal{H} = \pi \dot{\psi} + \vec{p} \cdot \vec{A} - \mathcal{L}$$

$$= \bar{\psi} i \gamma^0 \partial_0 \psi + \frac{E_1^2}{4\pi} - \bar{\psi} (i \gamma^0 \partial_0 + i \vec{\gamma} \cdot \nabla - m) \psi$$

$$- \frac{E_{11}^2}{8\pi} - \frac{E_1^2 - B^2}{8\pi} - e \psi^+ \Phi \psi + e \psi^+ \vec{\alpha} \cdot \vec{A} \psi$$

Note,

$$- e \psi^+ \Phi \psi = - \rho(\vec{x}) \Phi(\vec{x}),$$

$$- \frac{E_{11}^2}{8\pi} = - \frac{|\nabla \Phi|^2}{8\pi}.$$

$$e \psi^+ \vec{\alpha} \cdot \vec{A} \psi = e \bar{\psi} (\vec{\gamma} \cdot \vec{A}) \psi.$$

When we integrate over space we use the fact that

$$\int d^3 \vec{x} \frac{E_{11}^2}{8\pi} = \int d^3 \vec{x} \frac{|\nabla \Phi|^2}{8\pi} = - \int d^3 \vec{x} \frac{\Phi \nabla^2 \Phi}{8\pi}$$

$$= + \frac{1}{2} \int d^3 \vec{x} \rho \Phi$$

Thus the Hamiltonian can be written,

$$H = \int d^3 \vec{x} \mathcal{H} = H_D + H_{\text{em}} + H_{\text{int}}$$

where

$$H_D = \int d^3 \vec{x} \psi^+ (-i \vec{\alpha} \cdot \nabla + m \beta) \psi$$

$$H_{\text{em}} = \int d^3\vec{x} \frac{E_\perp^2 + B^2}{8\pi}$$

~~With $\vec{B} = \frac{1}{2} \vec{\nabla} \times \vec{A}$~~ "H-transverse"

$$H_{\text{int}} = H_{\text{coul}} + H_T,$$

where

$$H_{\text{coul}} = \frac{1}{2} \int d^3\vec{x} \rho \Phi = \frac{1}{2} \int d^3\vec{x} d\vec{x}' \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$= \frac{e^2}{2} \int d^3\vec{x} d^3\vec{x}' \frac{\psi^+(\vec{x}) \psi(\vec{x}) \psi^+(\vec{x}') \psi(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

and

$$H_T = e \int d^3\vec{x} \bar{\psi} (\vec{\gamma} \cdot \vec{A}) \psi.$$

We now quantize this, reinterpreting Φ, ψ, \vec{A} as field operators.

We must normal order so vacuum expectation values will vanish. We expand the free field Hamiltonians in normal modes, but leave the interacting Hamiltonians as spatial integrals. This gives

$$H = H_0 + H_1$$

$$H_0 = H_D + H_{\text{em}}$$

$$H_1 = H_{\text{coul}} + H_T$$

$$H_D = \int d^3\vec{x} : \psi(\vec{x})^+ (-i\vec{q} \cdot \vec{\nabla} + m\beta) \psi(\vec{x}) : = \sum_{ps} E (b_{ps}^+ b_{ps} + d_{ps}^+ d_{ps})$$

$$H_{\text{em}} = \int d^3\vec{x} : \frac{E_\perp^2 + B^2}{8\pi} : = \sum_{\lambda} \omega_{\lambda} a_{\lambda}^+ a_{\lambda}$$

$$H_{\text{coul}} = \frac{e^2}{2} \int d^3\vec{x} d^3\vec{x}' \frac{:\psi^+(\vec{x}) \psi(\vec{x}) \psi^+(\vec{x}') \psi(\vec{x}'):}{|\vec{x} - \vec{x}'|}$$

$$= \frac{1}{2} \int d^3\vec{x} d^3\vec{x}' \frac{:\rho(\vec{x}) \rho(\vec{x}'):}{|\vec{x} - \vec{x}'|}$$

$$H_T = e \int d^3\vec{x} : \bar{\Psi}(\vec{x}) \vec{\gamma} \cdot \vec{A}(\vec{x}) \Psi(\vec{x}):$$

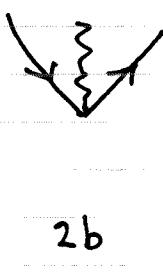
It is a guess that this is the correct Hamiltonian describing the electron-position-photon fields in interaction, but it's the simplest Hamiltonian consistent with Lorentz covariance.

Let us now examine the processes engendered by this $H_1 = H_{\text{coul}}$
(We start with H_T .)

+ H_T in 1st order TDPT. The general structure of a matrix element is

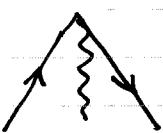
$$\langle f | H_T | i \rangle \sim \int d^3\vec{x} \langle f | : (b^\dagger \dots d^\dagger)(a \dots a^\dagger)(b \dots d^\dagger) : | i \rangle$$

There are 8 terms, which give the same Feynman diagrams shown on p.3, 5/5/06, except that a photon is attached to the vertex (either created or destroyed). Thus we have

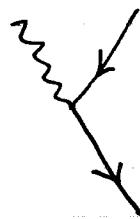




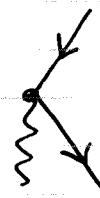
3a



3b



4a



4b.

These are the basic Feynman diagrams generated by one application of H_T . There are only 3-point vertices, because H_T is cubic in the field operators ($\bar{\psi} \psi A$). Also, charge is conserved at each vertex.

Return to $e^- e^+$ annihilation. Label initial and final modes, states.

$$(ps) \quad (p's') \quad \gamma \gamma' \\ e^- + e^+ \rightarrow \gamma \gamma'$$

initial state $|i\rangle = b_{ps}^+ d_{p's'}^+ |0\rangle$

final state $|f\rangle = |n\rangle = a_\gamma^+ a_{\gamma'}^+ |0\rangle$. This process cannot be accomplished by H_T in 1st order pertin theory, because H_T is capable of creating or destroying one photon, and we must create two photons on going from $|i\rangle$ to $|f\rangle$. But H_T can do it in 2nd order pertin theory. As for H_{Coul} , it cannot create any photons at all, since it has no \vec{A} in it. Therefore it does not contribute to pair annihilation at lowest order, and we ignore it henceforth.

Through 2nd order, TDPT gives the transition probability $i \rightarrow n$ as