throwing away the anticommutators themselves. This differs from what we did above in that we use d^+, d not c^+, c. Thus,

$$H = \int d^3\vec{x} : \psi^+ (-i\vec{\alpha}\cdot\nabla + m\beta)\psi : \; = \sum_{ps} E \left(b^+_{ps} b_{ps} + d^+_{ps} d_{ps}\right)$$

$$\vec{P} = \int d^3\vec{x} : \psi^+ (-i\nabla)\psi : \; = \sum_{ps} \vec{p} \left(b^+_{ps} b_{ps} + d^+_{ps} d_{ps}\right)$$

$$Q = \int d^3\vec{x} : \psi^+\psi : \; = -e\sum_{ps} \left(b^+_{ps} b_{ps} - d^+_{ps} d_{ps}\right).$$

Notice the sign changes. Now positrons have positive energy, momentum (and spin), but the charge operator is no longer positive definite. The quantity Dirac worked so hard to make positive definite in the 1st quantized theory is now replaced by an operator of either sign in the 2nd quantized theory. Energies are strictly positive, and we deal strictly with observable objects (electrons and positrons). Dirac's original goal of "curing" the "problems" with the KG eqn now appears as irrelevant, although there was no way to see that within the framework of the single particle (1st quantized) theory. The real conceptual breakthrough came with hole theory.

Now consider the Dirac electron field (2nd quantized) interacting with the EM field. Initially for simplicity we take the EM field as a specified given c-number field $A_\mu(\vec{x},t)$. So we don't need to worry about photons or Maxwell eqns.

To obtain the form of interaction, we go back to the free Dirac Lagrangian and use the minimal coupling prescription,

$$\mathcal{L} = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi, = \mathcal{L}_{free} \qquad \text{(free particle)}$$

$$i\partial_{\nu\mu} \to, \; i\partial_{\nu\mu} - qA_\mu, .$$

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int},$$

$$\mathcal{L}_{int} = -q\bar{\psi}\gamma^\mu A_\mu\psi = -q\bar{\psi}\not{A}\psi = -A_\mu J^\mu$$

where $J_\mu = q\bar{\psi}\gamma^\mu\psi$ is the current. Then for the Hamiltonian density we get

$$\mathcal{H} = \pi\dot{\psi} + \bar{\pi}\dot{\bar{\psi}} - \mathcal{L}$$
$$= \mathcal{H}_{free} + \mathcal{H}_{int},$$

(Notice $\mathcal{L}_{int}$ is a Lorentz scalar, as required for a Lorentz covariant theory.)

$$\mathcal{H}_{free} = \psi^+(-i\vec{\alpha}\cdot\nabla + m\beta)\psi$$

$$\mathcal{H}_{int} = -\mathcal{L}_{int} = q\bar{\psi}\gamma^\mu A_\mu\psi.$$

This is at the classical level. To quantize we reinterpret everything as operators and normal order. Thus the Hamiltonian is

$$H = \int d^3\vec{x} \; \mathcal{H} = H_{free} + H_{int} = H_0 + H_1,$$

where

$$H_0 = \int d^3\vec{x} \; : \psi^\dagger (-i\vec{\alpha}\cdot\nabla + m\beta)\psi : \; = \sum_{ps} E\big(\underbrace{b^\dagger_{ps} b_{ps}}_{\text{electrons}} + \underbrace{d^\dagger_{ps} d_{ps}}_{\text{positrons}} \big)$$

$$H_1 = q\int d^3\vec{x} \; : \bar{\psi} \, \gamma^\mu A_\mu \psi :$$

quantum fields

c-number field (for now).

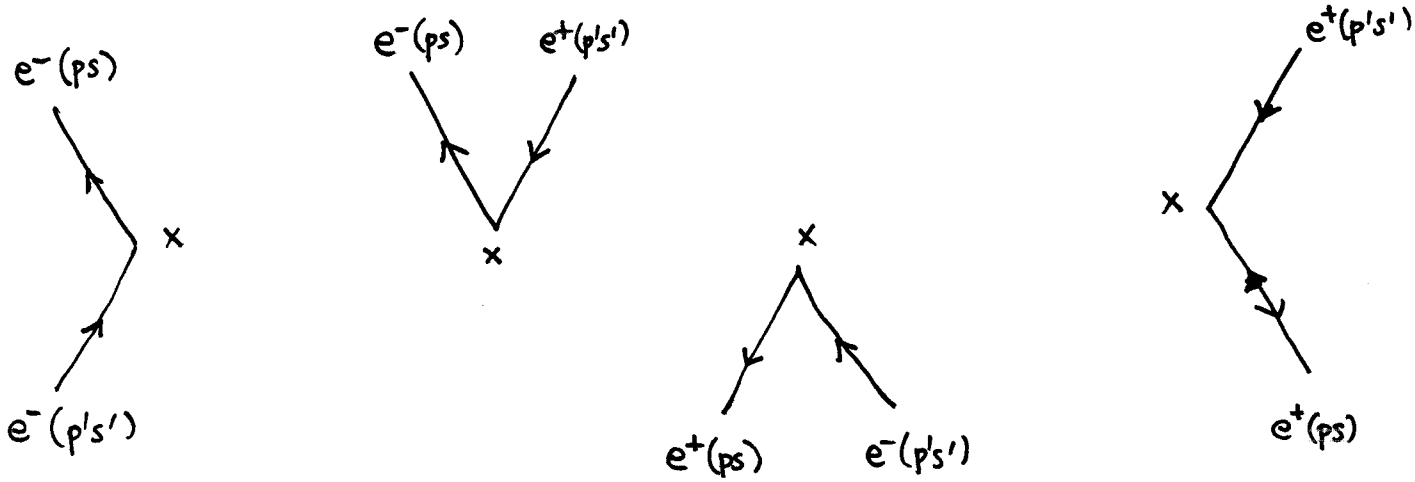
Let's begin by looking at the processes that can be engendered by H_1 in 1st order TDPT. The matrix element is $\langle f | H_1 | i \rangle$, and it has the general structure,

$$\langle f | H_1 | i \rangle \sim \int d^3\vec{x} \; \langle f | : \sum_{ps} (\ldots b^\dagger_{ps} \ldots d_{ps}) \; \gamma^\mu A_\mu(\vec{x},t) \sum_{p's'} (b_{p's'} \ldots d^\dagger_{p's'}) : | i \rangle$$

omitting all normalization consts etc. There are 4 types of terms that are formed from creation/annihil. operators, as shown in the table:

notice the sign change from normal ordering

$b^+_{ps}\, b_{p's'}$ $\qquad$ $b^+_{ps}\, d^+_{p's'}$ $\qquad$ $d_{ps}\, b_{p's'}$ $\qquad$ $-\, d^+_{p's'}\, d_{ps}$



(1) $\qquad$ (2) $\qquad$ (3) $\qquad$ (4)

Case (1) is scattering of an electron by the external field A_μ, indicated by the $\times$. Case (2) is creation of an e^+e^- pair in the field. Case (3) is the annihilation of an e^+e^- pair. Case (4) is the scattering of a positron. Arrows go in the direction of time for an electron, and backwards for a positron, following the ideas of Stückelberg and Feynman.

Now let's work through an example, the scattering of an electron in an electrostatic potential $\Phi(\vec{x})$. Thus,

$$A^\mu = (\Phi, \vec{0}),$$

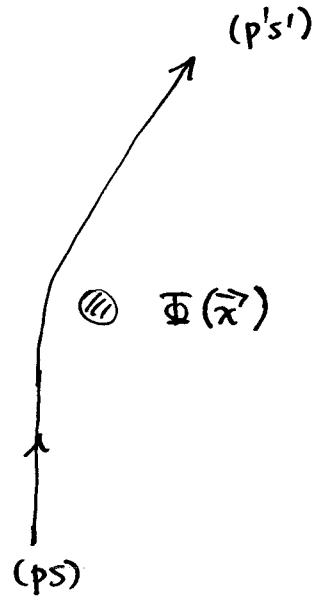
$$\gamma^\mu A_\mu = \not{A} = \gamma^0 \Phi(\vec{x}).$$

The initial and final states are

$$|i\rangle = b^+_{ps}|0\rangle$$

$$|n\rangle = b^+_{p's'}|0\rangle$$

an electron is state (ps) is scattered into state $(p's')$. The picture in 3D space is



It could be scattering by a nucleus (relativistic Rutherford scattering, also called <u>Mott scattering</u>.) The matrix element is (with $q = -e$ for electrons)

$$M = \langle n|H_1|i\rangle = \langle 0|b_{p's'}\ H_1\ b^+_{ps}|0\rangle$$

$$= -\frac{e}{V}\int d^3\vec{x}\ \langle 0|b_{p's'}:\sum_{p_1 s_1}\sqrt{\frac{m}{E_1}}\left(b^+_{p_1s_1}\bar{u}_{p_1s_1}e^{-i\vec{p}_1\cdot\vec{x}} + d_{p_1s_1}\bar{v}_{p_1s_1}e^{i\vec{p}_1\cdot\vec{x}}\right)$$

$$\times\ \gamma^0\Phi(\vec{x})\sum_{p_2s_2}\sqrt{\frac{m}{E_2}}\left(b_{p_2s_2}u_{p_2s_2}e^{i\vec{p}_2\cdot\vec{x}} + d^+_{p_2s_2}v_{p_2s_2}e^{-i\vec{p}_2\cdot\vec{x}}\right):$$

$$\times\ b^+_{ps}|0\rangle$$

Here we use the Fourier series for $\psi, \bar{\psi}$:

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left(b_{ps}\, u_{ps}\, e^{i\vec{p}\cdot\vec{x}} + d^{+}_{ps}\, v_{ps}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\bar{\psi}(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left(b^{+}_{ps}\, \bar{u}_{ps}\, e^{-i\vec{p}\cdot\vec{x}} + d_{ps}\, \bar{v}_{ps}\, e^{+i\vec{p}\cdot\vec{x}} \right).$$

Because of the anticommutation relations among the b's and d's, the only terms that survives in the two Fourier series in the matrix element M is the $b^{+}b$ term with $(p_1 s_1) = (p's')$ and $(p_2 s_2) = (ps)$. That is,

$$\langle 0 | b_{p's'}\, b^{+}_{p_1 s_1}\, \underbrace{b_{p_2 s_2}\, b^{+}_{ps}}_{\text{anticommute}} | 0 \rangle$$

$$= -\langle 0 | b_{p's'}\, b^{+}_{p_1 s_1}\, \overbrace{b^{+}_{ps}\, b_{p_2 s_2}}^{\text{kills.}} | 0 \rangle + \delta_{p_2 p}\, \delta_{s_2 s}\, \underbrace{\langle 0 | b_{p's'}\, b^{+}_{p_1 s_1} | 0 \rangle}_{\text{anticommute.}}$$

$$= -\langle 0 | b^{+}_{p_1 s_1}\, \overbrace{b_{p's'}\, | 0 \rangle}^{\text{kills}} + \delta_{p'p_1}\, \delta_{s's_1}.$$

All other terms $b^{+}d^{+}$ etc. give 0, i.e., only the electron scattering diagram (1) above contributes. Thus,

$$M = \frac{-e}{V} \int d^3\vec{x}\, \sqrt{\frac{m^2}{EE'}}\, \left(\bar{u}_{p's'}\, \gamma^0\, u_{ps} \right) e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}}\, \Phi(\vec{x})$$

$$= \frac{-e}{V}(2\pi)^{3/2}\sqrt{\frac{m^2}{EE'}}\;\tilde{\Phi}(\vec{q})\;(\bar{u}_{p's'}\gamma^0 u_{ps}),$$

where

$$\tilde{\Phi}(\vec{q}) = \int \frac{d^3\vec{x}}{(2\pi)^{3/2}}\, e^{-i\vec{q}\cdot\vec{x}}\,\Phi(\vec{x}), \qquad \text{F.T. of potential } \Phi(\vec{x}),$$

and where

$$\vec{q} = \vec{p}\,' - \vec{p} \qquad \text{(3-momentum transfer) in scattering.}$$

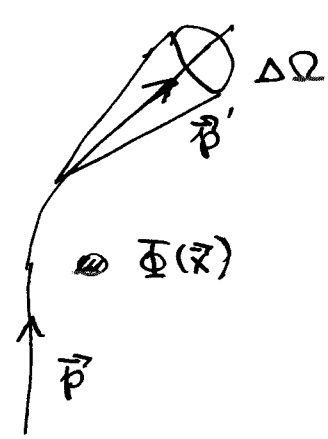
To get the diff. cross section, first compute the incident flux of electrons,

$$J_{inc} = \frac{1}{V}\,\frac{p}{E},$$

where $1/V$ = # inc. electrons / vol, velocity = p/E (relativistic expression).

Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{J_{inc}}\,\frac{1}{t}\,\frac{1}{\Delta\Omega}\sum_{\vec{p}\,'\in \text{cone}} |C_n(t)|^2$$



where

$$|C_n(t)|^2 = 2\pi t\;\Delta_t(E'-E)\;|M|^2$$

or,

$$\frac{d\sigma}{d\Omega} = V \frac{E}{p} \frac{1}{t} \frac{1}{\Delta\Omega} \boxed{\sum_{\vec{p}' \in \text{cone}}} 2\pi t \, \Delta_t(E'-E) \, \frac{e^2}{V^2} (2\pi)^3 \left|\tilde{\Phi}(\vec{q})\right|^2 \left(\frac{m^2}{EE'}\right) \left|\bar{u}_{p's'} \gamma^0 u_{ps}\right|^2$$

$$\lim_{V\to\infty} \quad \longrightarrow \quad \frac{V}{(2\pi)^3} \Delta\Omega \int_0^\infty p'^2 dp'$$

also $\lim_{t\to\infty} \Delta_t(E'-E) = \delta(E'-E) = \dfrac{\delta(p'-p)}{p/E}$. (magnitudes of $\vec{p}', \vec{p}$)

Clean it up, get

$$\frac{d\sigma}{d\Omega} = 2\pi e^2 m^2 \left|\tilde{\Phi}(\vec{q})\right|^2 \left|\bar{u}_{p's'} \gamma^0 u_{ps}\right|^2.$$

Note that when $p' = p$ (from δ-fn), we have $E' = E$, and the 4-momenta are

4-momenta. $p = (E, \vec{p})$

$p' = (E, \vec{p}')$

Now all we need to do is the contraction in Dirac spin space, $\bar{u}_{p's'} \gamma^0 u_{ps}$. The answer depends on the initial and final spin states s, s'. In the NR limit electron scattering by an electrostatic potential is independent of the spin, but at higher velocities the spin becomes more important.

For simplicity we will assume that the initial beam is

unpolarized and we don't care about the spin of the final electron. Then we must average over initial states and sum over final states, i.e.,

$$|\bar{u}_{p's'}\gamma^0 u_{ps}|^2 \rightarrow \frac{1}{2}\sum_{ss'} |\bar{u}_{p's'}\gamma^0 u_{ps}|^2$$

$$= \frac{1}{2}\sum_{ss'} \bar{u}_{p's'}\gamma^0 u_{ps}\,\bar{u}_{ps}\gamma^0 u_{p's'}$$

$$= \frac{1}{2}\sum_{ss'} \mathrm{tr}\left[\gamma^0 u_{ps}\bar{u}_{ps}\gamma^0 u_{p's'}\bar{u}_{p's'}\right]$$

$$= \frac{1}{2}\mathrm{tr}\left[\gamma^0\left(\sum_s u_{ps}\bar{u}_{ps}\right)\gamma^0\left(\sum_{s'} u_{p's'}\bar{u}_{p's'}\right)\right].$$

The two sums are projectors onto positive energy spinors, i.e.,

$$\Lambda_+(p) = \frac{\not{p}+m}{2m} = \sum_s u_{ps}\bar{u}_{ps}$$

$$\Lambda_-(p) = -\frac{\not{p}-m}{2m} = -\sum_s v_{ps}\bar{v}_{ps}$$

see p. 33 of Bjorken + Drell. So we must compute

$$\frac{1}{2}\mathrm{tr}\left[\gamma^0\left(\frac{\not{p}+m}{2m}\right)\gamma^0\left(\frac{\not{p}'+m}{2m}\right)\right].$$

Now we need some rules for taking the traces of products of γ matrices. First we present the rules, then we prove them.

<u>Rules:</u>

1. $\mathrm{tr}\, 1 = 4$
2. $\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$
3. $\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4 (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha})$
4. tr (any odd # γ's) $= 0$.

alternative versions of 2, 3:

2'. $\mathrm{Tr}\,(\not{a}\not{b}) = 4\, a\cdot b$

3'. $\mathrm{Tr}\,(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$

where a, b, c, d are 4 vectors and $a\cdot b = a^\mu b_\mu$ etc.

Proof of 1 is trivial.

Proof of 2:

$$\mathrm{tr}\,(\underbrace{\gamma^\mu \gamma^\nu}_{\text{anticomm.}}) = 2 g^{\mu\nu}\, \mathrm{tr}(1) - \underbrace{\mathrm{tr}(\gamma^\nu \gamma^\mu)}_{= \mathrm{tr}(\gamma^\mu \gamma^\nu) \;=\; \text{LHS.}}$$

cyclic perm.

So, $2\,\mathrm{tr}(\gamma^\mu \gamma^\nu) = 2 g^{\mu\nu} \times 4$, QED.

Proof of 3:

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 2g^{\mu\nu}\,\mathrm{tr}\,(\gamma^\alpha\gamma^\beta) - \mathrm{tr}\,(\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta)$$

$$= -2g^{\mu\alpha}\,\mathrm{tr}\,(\gamma^\nu\gamma^\beta) + \mathrm{tr}\,(\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta)$$

$$= +2g^{\mu\beta}\,\mathrm{tr}\,(\gamma^\nu\gamma^\alpha) - \mathrm{tr}\,(\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu)$$

$$= -\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = -\text{LHS}.$$

from which (3) follows. Obviously any trace of $2n$ γ's can be reduced to traces of $2n-2$ γ's.

Proof of 4 uses properties of $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, namely,

$$\gamma_5^2 = 1$$
$$\{\gamma_5, \gamma^\mu\} = 0.$$

Example of 3 γ's: 3 anticommutes.

$$\mathrm{tr}\,(\gamma^\mu\gamma^\alpha\gamma^\beta) = \mathrm{tr}\,(\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_5^2) = \mathrm{tr}\,(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma_5)$$

$$= -\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma_5^2) = -\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\alpha) = 0$$