Second quantizing the Dirac equation means reinterpreting the wave function ψ, which is supposed to represent the QM of a single particle (this is the first quantized version of the theory) as a quantum field. Thus, ψ goes from a c-number field into an operator field. The resulting quantum field theory is capable of handling problems with multiple electrons and positrons, including processes in which particles are created or destroyed.

To second quantize ψ, we follow the same general outline used earlier for quantizing the electromagnetic field. First we identify the classical field Hamiltonian (thinking of the first quantized ψ as a "classical" field) and the set of q's and p's that reproduce the desired field equations via classical Hamilton's equations. In the case of the 1st quantized Dirac field, these field equations are the Dirac equation,

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\nabla\psi + \beta m\psi \qquad \hbar = c = 1 \quad (\text{[redacted]}).$$

or

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

Then we replace the classical q's and p's with operators, obtaining a quantum field for ψ. The excitations of the field (the energy eigenstates) are then identified with particles. This follows what we did for photons.

To find the field Hamiltonian for the first quantized Dirac field ψ, we first find a Lagrangian. In field theory, the Lagrangian L is a spatial integral of a <u>Lagrangian density</u> $\mathcal{L}$,

$$L = \int d^3\vec{x}\ \mathcal{L}$$

where the integral represents the sum over the degrees of freedom of the field (one for each point of space). In simple applications, $\mathcal{L}$ is a function of the field (call it ψ in a general notation) and its first derivatives, $\mathcal{L} = \mathcal{L}(\psi, \partial_\mu \psi)$. This is like the Lagrangians $L(q, \dot{q})$ in ordinary particle mechanics. The action S is the time-integral of the Lagrangian,

$$S = \int dt\ L = \int d^4x\ \mathcal{L},$$

and by Hamilton's principle the equations of motion are equivalent to $\delta S = 0$ (S is stationary w.r.t. variations in the field ψ). This implies the <u>Euler-Lagrange equations</u> for the field,

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) = \frac{\partial \mathcal{L}}{\partial \psi}.$$

This is for a single field ψ; if there is more than one field (or if ψ has multiple components), then there is a separate

Euler - Lagrange eqn. for each field / component,

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\right) = \frac{\partial \mathcal{L}}{\partial \psi_a}, \qquad a = 1, 2, \ldots$$

Compare the E-L eqns in ordinary particle mechanics,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}, \qquad i = 1, 2, \ldots$$

If we wish the equations of motion to be relativistically covariant, then the action S must be a Lorentz scalar. Since the space-time volume element d^4x is a Lorentz scalar, this implies that the Lagrangian density $\mathcal{L}$ must be a Lorentz scalar, too. This imposes severe constraints on the form $\mathcal{L}$ is allowed to take in a covariant theory.

For example, for the free electromagnetic field, the simplest scalars that can be constructed from the field A^μ and its first derivatives are $F_{\mu\nu}F^{\mu\nu}$ and $A_\mu A^\mu$, of which only the first is gauge invariant. As it turns out, the $\mathcal{L}$ for the free EM field is

$$\mathcal{L}_{em} = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

where the $1/16\pi$ is conventional. If you add a term proportional to $A^\mu A_\mu$, it gives the photon ~~an~~ a nonzero mass (and breaks

gauge invariance).

For the free Dirac (first quantized or c-number) field, the simplest Lorentz scalars we can construct out of (the 4-component) ψ and its derivatives $\partial_\mu\psi$ are $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\partial_\mu\psi$. By taking linear combinations of these we find a $\mathcal{L}$ that works:

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \qquad (\hbar = c = 1).$$

We can check that this works. There are really 8 real fields, the real and imag parts of ψ_a, $a = 1,2,3,4$. Equivalently, we can treat ψ and ψ^+, or ψ and $\bar{\psi} = \psi^+\gamma^0$, as independent fields. The required derivatives are

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}i\gamma^\mu$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0.$$

The $\bar{\psi}$ equation immediately gives the Dirac eqn,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

The ψ equation does, too, in adjoint form:

$$\partial_\mu \left(\bar{\psi} i \gamma^\mu \right) = - m \bar{\psi}$$

$$\Rightarrow \quad (\partial_\mu \bar{\psi}) i \gamma^\mu = i (\partial_\mu \psi^\dagger) \gamma^0 \gamma^\mu = - m \psi^\dagger \gamma^0$$

$$\Rightarrow \quad i (\partial_\mu \psi^\dagger) \gamma^0 \gamma^\mu \gamma^0 + m \psi^\dagger = 0$$

$$\Rightarrow \quad -i \gamma^0 (\gamma^\mu)^\dagger \gamma^0 \partial_\mu \psi + m \psi = 0$$

$$\Rightarrow \quad i \gamma^\mu \partial_\mu \psi - m \psi = 0.$$

So $\mathcal{L}_D$ is an acceptable field Lagrangian for the free Dirac field.

Now we need the field Hamiltonian. In ordinary particle mechanics we define the canonical momentum by

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

and then the Hamiltonian by

$$H = \sum_i p_i \dot{q}_i - L.$$

In field theory, we define the <u>momentum fields</u> π_a by

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a}, \quad \text{where } \dot{\psi}_a = \frac{\partial \psi_a}{\partial t},$$

and then the <u>Hamiltonian density</u> by

$$\mathcal{H} = \sum_a \pi_a \dot{\psi}_a - \mathcal{L}.$$

Finally, the Hamiltonian is

$$H = \int d^3\vec{x} \quad \mathcal{H}.$$

To apply this to the Dirac field, write the Lagrangian as

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot\nabla - m\right)\psi .$$

Then

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0$$

$$\bar{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

so

$$\mathcal{H} = \pi\dot{\psi} + \bar{\pi}\dot{\bar{\psi}} - \mathcal{L} = i\bar{\psi}\gamma^0\frac{\partial\psi}{\partial t} - \mathcal{L}$$

$$= \bar{\psi}\left(-i\vec{\gamma}\cdot\nabla + m\right)\psi$$

$$= \psi^+\left(-i\vec{\alpha}\cdot\nabla + m\beta\right)\psi .$$

The Hamiltonian density is $\psi^+\psi$ sandwiched around the usual Dirac (quantum) Hamiltonian operator. Thus, the field Hamiltonian is

$$H = \int d^3\vec{x} \quad \psi^+\left(-i\vec{\alpha}\cdot\nabla + m\beta\right)\psi$$

It is the expec. val. of the qu. Hamiltonian w.r.t. the state ψ.

$\hbar = c = 1$ in the following.

Compare the field Hamiltonian for the (1st quantized) Dirac field with that of the EM field:

$$H = \int d^3\vec{x}\ \psi^+ (-i\vec{\alpha}\cdot\nabla + m\beta)\psi \qquad \text{vs.} \qquad H = \int d^3\vec{x}\ \frac{E^2+B^2}{8\pi}.$$

These are classical Hamiltonians in the sense that when the right q's and p's are identified, then the correct eqns. of motion follow from the classical Hamilton's eqns. In the case of the Dirac eqn, the correct eqns of motion are

$$i\frac{\partial\psi}{\partial t} = (-i\vec{\alpha}\cdot\nabla + m\beta)\psi$$

and in the case of the EM field they are Maxwell's eqns. The q's and p's can be found systematically by proceeding from the Lagrangian, but it's easier just to guess them (and then check the answers). We do this by developing the normal mode expansion of the Dirac (1st quantized) field.

The normal mode expansion of the classical EM field is a representation of $\vec{A}$ as a lin. comb. of plane light waves,

$$\vec{A}(\vec{x}) = \sqrt{\frac{2\pi}{V}} \sum_{\vec{k}\mu} \frac{1}{\sqrt{\omega}} \left(a_{\vec{k}\mu}\, \hat{\varepsilon}_{\vec{k}\mu}\, e^{i\vec{k}\cdot\vec{x}} + \text{c.c.} \right).$$

For the (1st quantized) Dirac field, it is a representation of $\psi(\vec{x})$ (the 4-comp. spinor) as a lin. comb. of free particle (plane

wave) solutions. These are

$$u(p,s)\, e^{i\vec{p}\cdot\vec{x}} \qquad \text{(pos. energy)}$$

$$\text{and} \quad v(p,s)\, e^{-i\vec{p}\cdot\vec{x}} \qquad \text{(neg. energy)}.$$

These are parameterized by a 3-momentum $\vec{p}$, we define $E = \sqrt{m^2 + |\vec{p}|^2}$ (the pos. square root) and $p = (E, \vec{p})$ the 4-vector. Both E, p are determined once $\vec{p}$ is given. s is the spin 4-vector which takes on only 2 values ($\pm$ some direction in the rest frame). The energy and momentum eigenvalues of the pos. energy solns are $E, \vec{p}$, and of the neg. energy solns, $-E, -\vec{p}$. For given $\vec{p}$ there are 4 solns ($\pm s$, u and v). The spinors u, v satisfy the Hermitian orthonormality conditions,

$$u(p,s)^\dagger u(p,s') = \frac{E}{m}\delta_{ss'}$$

$$u(p,s)^\dagger v(\tilde{p},s') = 0$$

$$v(p,s)^\dagger u(\tilde{p},s') = 0$$

$$v(p,s)^\dagger v(p,s') = \frac{E}{m}\delta_{ss'}$$

where $\tilde{p} = (E, -\vec{p})$.

We normalize the free particle solns above in a box w. vol. $V = L^3$:

Dirac's prob. density.

$$\int_{box} d^3\vec{x}\; \psi^\dagger\psi = 1$$

This means

integer vector

$$\vec{p} = \frac{2\pi}{L}\vec{n} = \underline{\text{discrete}}$$

Thus, including normalization, the free particle solutions are

$$\frac{1}{\sqrt{V}}\sqrt{\frac{m}{E}}\; u(p,s)\; e^{i\vec{p}\cdot\vec{x}}$$

$$\frac{1}{\sqrt{V}}\sqrt{\frac{m}{E}}\; v(p,s)\; e^{-i\vec{p}\cdot\vec{x}}$$

An arbitrary Dirac wave fn ψ can be represented as a lin. comb. of these solutions,

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}}\sum_{ps}\sqrt{\frac{m}{E}}\left(b_{ps}\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

where b_{ps}, c_{ps} are the expansion coefficients. This is because the free particle Dirac Hamiltonian is Hermitian and complete. The p in the $\sum_p$ means really $\sum_{\vec{p}}$, and $\sum_s$ means sum over 2 values of s corresp. to $\pm\hat{s}$ in the rest frame. We call b_{ps} and c_{ps} the <u>mode amplitudes</u>. They are (in the 1st quantized Dirac theory) the probability amplitudes to find the state ψ in a given free particle state of + or − energy with mode ps $(= \vec{p}, s)$.

Let us now express the field Hamiltonian in terms of the mode amplitudes. Recall when we did this for the free EM field, we obtained

$$H = \int d^3\vec{x}\; \frac{E^2+B^2}{8\pi} = \sum_{\vec{k}\mu} \hbar\omega\, |a_{\vec{k}\mu}|^2 .$$

For the Dirac field, we get

$$H = \int_{box} d^3\vec{x} \;\; \psi^\dagger \left(-i\vec{\alpha}\cdot\nabla + m\beta\right)\psi$$

$$= \int d^3\vec{x} \; \frac{1}{V} \sum_{\substack{ps \\ p's'}} \sqrt{\frac{m^2}{EE'}} \left(b^*_{p's'} u^\dagger(p's') e^{-i\vec{p}'\cdot\vec{x}} + c^*_{p's'} v(p's') e^{+i\vec{p}'\cdot\vec{x}} \right)$$

$$\times \left(-i\vec{\alpha}\cdot\nabla + m\beta\right)\left(b_{ps}\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

When the Dirac Hamiltonian $(-i\vec{\alpha}\cdot\nabla + m\beta)$ acts on free particle solns, it brings ~~out~~ out either E or $-E$ (for pos + neg. energy solns). Thus the last 2 factors become

$$E \left(b_{ps}\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} - c_{ps}\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

↑ notice sign.

So the whole expression for H involves 4 major terms, call them uu, uv, vu and vv terms.

$$uu\text{-term} = \frac{1}{V}\int d^3\vec{x} \sum_{\substack{ps \\ p's'}} \sqrt{\frac{m^2}{EE'}}\, \overset{E}{\downarrow}\, b^*_{p's'}\, b_{ps}\; u(p's')^\dagger u(ps)\; e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}$$

$$= \sum_{\substack{ps \\ p's'}} \sqrt{\frac{m^2}{EE'}}\; E\; b^*_{p's'}\, b_{ps}\; u(p's')^\dagger u(ps)\; \delta_{\vec{p},\vec{p}'}$$

$$= \sum_{pss'} \frac{m}{E} E \; b^*_{ps'} b_{ps} \underbrace{u(ps')^\dagger u(ps)}_{\delta_{ss'} \frac{E}{m}}$$

since $E' = E$
$p' = p$
when $\vec{p}' = \vec{p}$

$$= \sum_{ps} E \; |b_{ps}|^2 .$$

Sim. we find the uv and vu terms vanish, and the vv term gives

$$-\sum_{ps} E \; |c_{ps}|^2 .$$

Thus,

$$H = \sum_{ps} E \left(|b_{ps}|^2 - |c_{ps}|^2 \right)$$

The calculation above is really just a check of the orthonormality of the free particle solutions in a box. The answer is obvious in the 1st quantized theory, since $|b_{ps}|^2$ and $|c_{ps}|^2$ are just the probabilities of finding $\psi(x)$ in the given (pos or neg energy) free particle state, which have energy $\pm E$. That is, $H_{\substack{cl \\ field}} = \langle \psi | H_{Dirac} | \psi \rangle$.

The mode expansion above for $\psi(\vec{x})$ is taken at a fixed time, say, $t=0$. If we regard $\psi(\vec{x})$ as an initial condition for the free particle Dirac eqn, then the subsequent time evolution is easy to write down, because the plane waves in the normal modes

have a trivial time evolution. That is, if

$$\psi(\vec{x},0) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left[b_{ps}(0)\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}(0)\, v(p,s)\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

then

$$\psi(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left[b_{ps}(t)\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}(t)\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

where

$$b_{ps}(t) = b_{ps}(0)\, e^{-iEt}$$
$$c_{ps}(t) = c_{ps}(0)\, e^{+iEt}.$$

Equivalently,

$$\psi(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left[b_{ps}(0)\, u(ps)\, e^{-ip_\mu x^\mu} + c_{ps}(0)\, v(ps)\, e^{+ip_\mu x^\mu} \right].$$

The t-evolution of b_{ps} or c_{ps} (the mode amplitudes) in the complex plane is a circle (clockwise for b_{ps}, counterclockwise for c_{ps}). This looks like the evolution of a harmonic oscillator in the phase (q p) plane, and suggests that the real and imag. parts of the b_{ps} and c_{ps} are the q's and p's of the system. It is easily checked that this is correct, i.e., with this choice of q's and p's, Hamilton's (classical) eqns reproduce the correct time evolution.

This completes the classical field description of the 1st quantized Dirac theory. Now as we did with the EM field we quantizes this field by

reinterpreting the q's and p's as operators satisfying the canonical commutation relations. This is Dirac's prescription for converting a classical system into a quantum one, and it is a guess that must be checked by the consistency of the quantum theory and its comparison with experiment. In the present case, it means that the mode amplitudes b_{ps}, c_{ps} become operators that satisfy the commutation relations,

$$[b_{ps}, b^+_{p's'}] = [c_{ps}, c^+_{p's'}] = \delta_{pp'}\delta_{ss'}$$

and all other commutators vanish,

$$[b_{ps}, b_{p's'}] = [b_{ps}, c_{p's'}] = [b_{ps}, c^+_{p's'}]$$
$$= [b^+_{ps}, c_{p's'}] = [b^+_{ps}, c^+_{p's'}] = [c_{ps}, c_{p's'}] = [c^+_{ps}, c^+_{p's'}] = 0.$$

Now the wave field $\psi(\vec{x})$ is a quantum field,

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left(b_{ps}\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}} \right),$$

now operators

and the Hamiltonian H is a field operator. To obtain a field Hamiltonian with vanishing vacuum expectation value, we normal order the expression for H, which means move all b^+'s and c^+'s to the left, all b's and c's to the right, and discard any commutators.

For H this gives,

$$H = \int d^3\vec{x} \; : \psi^+ \left(-i\vec{\alpha}\cdot\nabla + m\beta \right) \psi :$$

$$= \sum_{ps} E \left(b^+_{ps} b_{ps} - c^+_{ps} c_{ps} \right) .$$

where : : means, "normal order". Here there is no normal ordering to be done, since $\psi^+ \sim b^+, c^+$ is already on the left and $\psi \sim b, c$ on the right.

As yet we have little idea about what this quantum field means physically, but on the basis of our experience with photons we can expect b^+_{ps}, (c^+_{ps}) to create electrons with pos (neg) energy and b_{ps}, c_{ps} to destroy them. The theory will clearly be a multiparticle theory, since we can create as many particles as we like. Recall that even the first quantized Dirac theory became a multiparticle theory when Dirac introduced his sea to overcome difficulties with the negative energy solutions.

There is one major problem with what we have done so far, however. Following photon formalism, let's create a 2-electron state with electrons in modes ps and p's'. This is either

$$b^+_{ps}\, b^+_{p's'} |0\rangle \quad \text{or} \quad b^+_{p's'}\, b^+_{ps} |0\rangle .$$

since $[b^+_{ps}, b^+_{p's'}] = 0$, these two states are identical, but since they differ by an exchange of electron labels and since electrons are fermions, they should differ by a sign. In other words, our field theory does not satisfy Fermi-Dirac statistics. The difficulty traces back to the assumed commutation ~~relations~~ relations of the b's and c's.

To fix this let us postulate instead that the b's and c's satisfy <u>anticommutation</u> <u>relations</u>, i.e.,

$$\{b_{ps}, b^+_{p's'}\} = \{c_{ps}, c^+_{p's'}\} = \delta_{pp'}\delta_{ss'}$$

all other anticommutators vanish,

$$\{b_{ps}, b_{p's'}\} = \dots = \{c^+_{ps}, c^+_{p's'}\} = 0.$$

Then we have

$$b^+_{ps}\, b^+_{p's'}\, |0\rangle = -\, b^+_{p's'}\, b^+_{ps}\, |0\rangle$$

and all is well with the symmetrization postulate. This does mean, however, that our b's and c's will satisfy different properties from the a's that were used in photon theory. (of course, photons are bosons, so the commutation relations were correct for them.).

For bosons, we can put any number of particles in a mode. We have

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n\, |0\rangle$$

where we suppress indices but one mode is referred to.

For electrons, we put one (+ energy) electron in mode ps with

$$b^+_{ps}|0\rangle$$

but if we try to put two in the same mode we get

$$b^+_{ps}\, b^+_{ps}\, |0\rangle = 0$$

since

$$(b^+_{ps})^2 = \underbrace{b^+_{ps}\, b^+_{ps}}_{\text{anti commute}} = -\, b^+_{ps}\, b^+_{ps} = 0.$$

Similarly, we find $b^2_{ps} = 0$. This of course is the Pauli principle: we can have either 0 or 1 electron in a mode, no more. Similarly, define the number operator, (for + energy electrons)

$$N_{ps} = b^+_{ps}\, b_{ps}.$$

Then

$$N^2_{ps} = b^+_{ps}\, b_{ps}\, \underbrace{b^+_{ps}}_{\text{anti comm.}}\, b_{ps} = -\, b^+_{ps}\, b^+_{ps}\, b_{ps}\, b_{ps} + b^+_{ps}\, b_{ps}$$

$$= 0 + N_{ps},$$

or $\quad N^2_{ps} - N_{ps} = N_{ps}\,(N_{ps} - 1) = 0.$

Thus the eigenvalues of N_{ps} are either 0 or 1.

The Hamiltonian can be written in terms of number operators,

$$H = \sum_{ps} E \left(N_{ps}^{(+)} - N_{ps}^{(-)} \right).$$

where

$$N_{ps}^{(+)} = b_{ps}^{+} b_{ps} \qquad (+\text{ energy})$$

$$N_{ps}^{(-)} = c_{ps}^{+} c_{ps} \qquad (-\text{ energy}).$$

We must do some work to interpret and understand the new field theory. Let's begin by computing the Heisenberg eqns of motion for the b's and c's. This gives

$$\dot{b}_{ps} = -i \left[b_{ps}, H \right]$$

$$= -i \sum_{p's'} E' \left(\underbrace{\left[b_{ps}, b_{p's'}^{+} b_{p's'} \right]} + \underbrace{c\text{-term}}_{\text{vanishes.}} \right)$$

$$= \underbrace{b_{ps} b_{p's'}^{+}}_{\text{anticomm.}} b_{p's'} - b_{p's'}^{+} b_{p's'} b_{ps}$$

$$= - b_{p's'}^{+} \underbrace{b_{ps} b_{p's'}} - b_{p's'}^{+} b_{p's'} b_{ps} + \delta_{pp'} \delta_{ss'} b_{p's'}$$

$$= + b_{p's'}^{+} \cancel{b_{p's'} b_{ps}} - \cancel{b_{p's'}^{+} b_{p's'}} b_{ps} + \delta_{pp'} \delta_{ss'} b_{p's'}$$

so $\dot{b}_{ps} = -i E b_{ps}$, or $b_{ps}(t) = b_{ps}(0)\, e^{-iEt}$.

Sim. we find $c_{ps}(t) = c_{ps}(0)\, e^{+iEt}$. These are the

same eqns and solutions we found in the classical theory (now reinterpreted as operators). Thus the quantum field $\psi(\vec{x},t)$ evolves in time by the same formula above quoted in the 1st quantized theory. (And the Dirac eqn. becomes the Heisenberg eqn. of motion for the quantum field ψ.)

The eigenstates of the Hamiltonian are specified by a string of occupation numbers, one for each mode, $(\ldots n^{\pm}_{ps} \ldots)$ or $\{n^{\pm}_{ps}\}$, where each $n^{\pm}_{ps} = 0$ or 1, and

$$|\ldots n^{\pm}_{ps} \ldots\rangle = \prod_{ps} (b^{+}_{ps})^{n^{+}_{ps}} \prod_{ps} (c^{+}_{ps})^{n^{-}_{ps}} |0\rangle$$

where the $\pm$ on $n^{\pm}_{ps}$ means # of pos. or neg. energy electrons.

Allowing H to act on these states, we get

$$H|\ldots n^{+}_{ps} \ldots n^{-}_{ps} \ldots\rangle = \sum_{ps} E\,(n^{+}_{ps} - n^{-}_{ps})\,|\ldots n^{+}_{ps} \ldots n^{-}_{ps} \ldots\rangle .$$

For simplicity just look at a single electron state, say $b^{+}_{ps}|0\rangle$. Then

$$H\left(b^{+}_{ps}|0\rangle\right) = E\left(b^{+}_{ps}|0\rangle\right) \qquad \text{where } E = E(\vec{p}).$$

sim. $$H\left(c^{+}_{ps}|0\rangle\right) = -E\left(c^{+}_{ps}|0\rangle\right).$$

The excitations have energy $E = E(\vec{p})$ (or $-E$).

What is their momentum? For this we need a field momentum

operator. For this we go back to the 1st quantized theory and derive a conserved momentum vector $\vec{P}$ for the Dirac field. This follows by applying Noether's theorem to the field Lagrangian $\mathcal{L}$, which is invariant under translations. We find

$$\vec{P} = \int d^3\vec{x} \quad \psi^+ \, (-i\nabla)\psi .$$

This is simple: it is just the expectation value (in the 1st qu. theory) of the momentum operator $-i\nabla$. Now we quantize (i.e. 2nd quantize) ψ to get the field operator $\vec{P}$. We must normal order. We also express in terms of b's and c's. This gives

$$\vec{P} = \int d^3\vec{x} \; : \psi^+ (-i\nabla)\psi : \; = \sum_{ps} \vec{p} \left(b^+_{ps} b_{ps} - c^+_{ps} c_{ps} \right).$$

Thus

$$\vec{P} \left(b^+_{ps} |0\rangle \right) = \vec{p} \left(b^+_{ps} |0\rangle \right)$$

$$\vec{P} \left(c^+_{ps} |0\rangle \right) = -\vec{p} \left(c^+_{ps} |0\rangle \right).$$

The excitations $b^+_{ps}|0\rangle$ have energy E and momentum $\vec{p}$, while $c^+_{ps}|0\rangle$ has energy + mom. $-E$ and $-\vec{p}$. With this we are satisfied that the excitations should be identified with electron states.

We have worked so far with two bilinear quantities of the 1st qu. classical field, H and $\vec{P}$, which we carried over to field operators. These

is another bilinear quantity important in the 1st quantized theory, namely the total probability:

$$1 = \int d^3\vec{x} \; \psi^+\psi$$

When we 2nd quantize ψ this becomes a field operator,

$$\int d^3\vec{x} \; :\psi^+\psi: \; = \sum_{ps} \left(b^+_{ps} b_{ps} + c^+_{ps} c_{ps} \right).$$

It is the sum of + and − energy number operators, so it represents the total # of electrons in the system (either pos or neg. energy). If we multiply by $q = -e$ we get a <u>charge operator</u>,

$$Q = -e \int d^3\vec{x} \; :\psi^+\psi: \; = -e \sum_{ps} \left(b^+_{ps} b_{ps} + c^+_{ps} c_{ps} \right)$$

At this point we have a 2nd quantized version of Dirac's theory of pos and neg. energy electrons. It gives the correct FD statistics in multiparticle problems, but otherwise its physical content does not go beyond what we had with the first quantized theory. In particular, it does not address the interpretational difficulties of the neg. energy solutions.

To fix this up, we borrow ideas from hole theory. If all the negative energy states are filled, as Dirac supposed, then when we

excite an electron out of a negative energy state, we create a hole. Thus, the <u>destruction</u> of a neg. energy electron in the sea is equivalent to the <u>creation</u> of a hole (or positron). Thus let us define

$$c_{ps} = d^{+}_{ps}$$

where d^{+}_{ps} creates a position of charge, energy, momentum and spin $+e, +E, +\vec{p}$ and $+s$. Likewise, if an electron makes a radiative transition to an unoccupied neg. energy state (a hole), it <u>creates</u> a neg. energy electron or <u>destroys</u> a hole. So let us write

$$c^{+}_{ps} = d_{ps}.$$

Note that with these defn's, the d_{ps}, d^{+}_{ps} satisfy anticomm. relations exactly like the b's and c's,

$$\{d_{ps}, d^{+}_{p's'}\} = \delta_{pp'}\delta_{ss'}$$

all other $\{\ ,\ \} = 0$.

Now the quantum field is

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left(b_{ps}\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + d^{+}_{ps}\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}} \right).$$

Also, the field Hamiltonian becomes,

$$H = \sum_{ps} E \left(b^+_{ps} b_{ps} - d_{ps} d^+_{ps} \right).$$

Notice the ordering of the d operators. If we anticommute them, we get

$$H = \sum_{ps} E \left(b^+_{ps} b_{ps} + d^+_{ps} d_{ps} \right) - \sum_{ps} E$$

The final term is ∞. One interpretation is that it is the ∞ (neg.) energy of the filled Dirac sea. Another interp. is that it is an ∞ term resulting from the ordering ambiguities on passing from the classical expression to a qu. operator. That is, it is a zero point term.

We throw away zero point terms to make vacuum expectation values vanish. But is the vacuum the state without any electrons of any kind, pos or neg. energy, or is it the state without either electrons or positrons? The latter is more physical, and if we want $\langle 0|H|0\rangle = 0$ for this vacuum, then we must throw away the term $-\sum_{ps} E$ above.

This leads to a new interpretation of normal ordering: We migrate b^+'s and d^+'s to the left, b's and d's to the right, keeping any sign changes on applying anticommutators, but