

Now we turn to the optical theorem. First, an almost one-line proof for central force potentials, using the method of phase shifts. We have:

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos \theta)$$

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l.$$

Evaluate  $f(\theta)$  in the forward direction,  $\theta=0$ , where  $P_l(1)=1$ :

$$f(0) = \frac{1}{k} \sum_l (2l+1) \sin \delta_l (\cos \delta_l + i \sin \delta_l),$$

so

$$\frac{4\pi}{k} \operatorname{Im} f(0) = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l = \sigma.$$

In summary,

$$\sigma = \frac{4\pi}{k} \operatorname{Im} f(0)$$

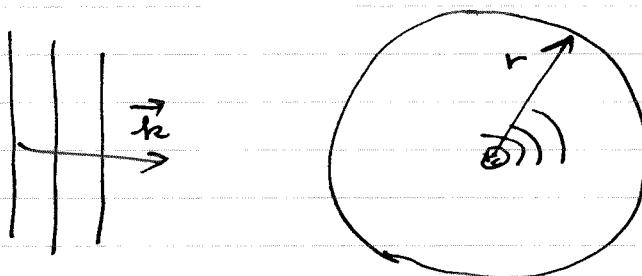
(Optical theorem).

This is easy, but it doesn't give any insight into the meaning.  
(The total cross section is proportional to the  $\operatorname{Im}$  part of the scattering amplitude in the forward direction).

Now a longer proof that carries more insight. Also generalizes to non-central force potentials. We consider the probability current,

$$\vec{J} = \frac{-i\hbar}{2m} \psi^* \nabla \psi + \text{c.c.}$$

- in QM (scalar particle). For a scattering soln.  $H|\psi\rangle = E|\psi\rangle$   
it satisfies  $\nabla \cdot \vec{J} = 0$ , hence its integral over any closed surface  
is 0. We will integrate it over a large sphere of radius  $r$   
centered on the scatterer, and take  $r \rightarrow \infty$ .



so,

$$\sigma = \int_{\text{sphere}} \vec{J} \cdot d\vec{a} = r^2 \int J_r d\Omega \quad J_r = \hat{\vec{r}} \cdot \vec{J}.$$

Since we are in the asymptotic regime, we can take the asymptotic form of  $\psi$ ,

$$\psi(\vec{r}) = A \left[ e^{i\vec{k} \cdot \vec{r}} + f(\theta, \phi) \frac{e^{i k r}}{r} \right] + \text{higher order terms.}$$

↑  
normaliz. const.

With  $\vec{k} = k\hat{z}$ , we have  $\vec{k} \cdot \vec{r} = kz = kr \cos\theta$ . Then

$$\vec{J} = \frac{|A|^2 v}{2} \left[ e^{-ikr \cos\theta} + f^* \frac{e^{-ikr}}{r} \right] \left[ e^{ikr \cos\theta} \hat{z} + \frac{f}{r} e^{ikr} \hat{r} \right] + \text{c.c.},$$

where we drop terms that  $\rightarrow 0$  faster than  $1/r^2$ , since they won't contribute to the integral.  $\vec{J}$  breaks up into the incident, scattered, and cross-term ( $x$ ) or interference currents,  $\vec{J}_{\text{inc}}$ ,  $\vec{J}_{\text{sc}}$ ,  $\vec{J}_x$ . Here

$$\vec{J}_{\text{inc}} = \frac{|A|^2 v}{2} \hat{z} = \text{const.}$$

so

$$\int_{\text{sphere}} \vec{J}_{\text{inc}} \cdot d\vec{a} = 0.$$



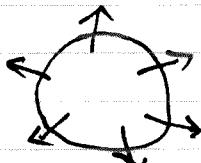
The incident flux goes right through the sphere. As for  $\vec{J}_{\text{sc}}$ ,

$$\vec{J}_{\text{sc}} = |A|^2 v \frac{|f|^2}{r^2} \hat{r}, \quad |f|^2 = \frac{d\sigma}{d\Omega}$$

so  $\int_{\text{sphere}} \vec{J}_{\text{sc}} \cdot d\vec{a} = |A|^2 v \int d\Omega \frac{d\sigma}{d\Omega} = |A|^2 v \sigma.$

This is the positive flux of scattered particles.

of course it's proportional to  $\sigma$ ; this is the def. of  $\sigma$ .



The scattered particles are coming out of the sphere, so there must be some net flux in. It must come from the cross terms.

$$\vec{J}_x = \frac{|A|^2 v}{2} \left\{ e^{ikr(1-\cos\theta)} \frac{f}{r} \hat{r} + e^{-ikr(1-\cos\theta)} \frac{f^*}{r} \hat{z} \right.$$

$$\left. + e^{-ikr(1-\cos\theta)} \frac{f^*}{r} \hat{r} + e^{ikr(1-\cos\theta)} \frac{f}{r} \hat{z} \right\}$$

$$= \frac{|A|^2 v}{2} e^{ikr(1-\cos\theta)} \frac{f}{r} (\hat{r} + \hat{z}) + \text{c.c.}$$

so,

$$= r^2 \int d\Omega (\hat{r} \cdot \vec{J}_x)$$

$$\int_{\text{sphere}} \vec{J}_x \cdot d\vec{\alpha} = \frac{|A|^2 v}{2} r \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta e^{ikr(1-\cos\theta)} f(\theta, \phi) (1 + \cos\theta) + \text{c.c.}$$

where we use  $(\hat{r} + \hat{z}) \cdot \hat{r} = 1 + \cos\theta$ . We want this in the limit  $r \rightarrow \infty$ , which is tricky since the integral is multiplied by  $r$  but the integrand oscillates ever more rapidly as  $r \rightarrow \infty$ . It helps to clarify this if we ~~not~~ integrate by parts, using

$$\sin\theta d\theta e^{ikr(1-\cos\theta)} = -\frac{1}{ikr} d e^{ikr(1-\cos\theta)}$$

so,

$$\int_{\text{sphere}} \vec{J}_x \cdot d\vec{\alpha} = \frac{|A|^2 v}{2} \frac{r}{ikr} \int_0^{2\pi} d\phi \left\{ e^{ikr(1-\cos\theta)} f(\theta, \phi) (1 + \cos\theta) \right\}_0^\pi$$

$$- \int_0^\pi d\theta e^{ikr(1-\cos\theta)} \frac{\partial}{\partial \theta} [f(\theta, \phi)(1 + \cos\theta)] \} + \text{c.c.}$$

The  $r$ 's out front cancel and the remaining integral still has the oscillating integrand, so it goes to 0 as  $r \rightarrow \infty$ , and we just have the 1st term,

$$e^{i(kr)(1-\cos\theta)} f(\theta, \phi) (1+\cos\theta) \Big|_0^\pi = 0 - 2f(0, \phi).$$

But  $f(0, \phi)$  is  $f$  at the north pole, where it doesn't depend on  $\phi$ . Just call it  $f(0)$ . So the  $\phi$  integral can be done, and gives  $2\pi$ . Thus,

$$\begin{aligned} \int_{\text{sphere}} \vec{J}_x \cdot d\vec{a} &= \frac{|A|^2 v}{2} \frac{2\pi}{k} \left[ -\frac{2f(0)}{i} + \frac{2f(0)^*}{i} \right] \\ &= -|A|^2 v \frac{4\pi}{k} \Im f(0). \end{aligned}$$

Putting it together,  $\int \vec{J} \cdot d\vec{a} = 0 \Rightarrow$

$$\boxed{\sigma = \frac{4\pi}{k} \Im f(0)}$$

We see that the scattered flux is compensated by an inward coming flux from down the  $+z$ -axis, coming from the interference between the incident and scattered wave. This flux cancels out some of the incident flux, creating the shadow downstream from the scatterer. More later...