Physics 221B
Spring 2012
Homework 25
Due Friday, April 27, 2012

Reading Assignment:

Please read the lectures notes for the lectures of April 9 to April 20, including the section from Sakurai on hole theory. These two weeks of lectures covered hole theory, the second quantization of the Dirac equation, and an application of first order perturbation theory using the electron-positron field interacting with an external electrostatic field (electron scattering in a potential). This homework concerns another application that can be accomplished with second quantized fermion fields and first order perturbation theory. The lectures also got started on an application of second order perturbation theory, namely, electron-positron annihilation, which we will continue next week.

1. We use natural units ($\hbar = c = 1$) in this problem.

Beta decay is the reaction

$$n \to p + e^- + \bar{\nu}, \tag{1}$$

the decay of a neutron into a proton, an electron, and an antineutrino. The Feynman diagram for this process is shown in Fig. 1.

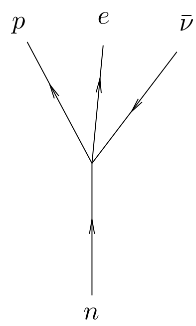

Fig. 1. Feynman diagram for β-decay.

In 1934 Fermi wrote down a field theory to explain β-decay. Since that time our understanding of weak interaction physics has become much more sophisticated (parity violation, helicity of neutrinos, weak currents, three types of neutrinos, electroweak unification, W and Z bosons, neutrino oscillations and neutrino mass, etc etc) but Fermi's theory does explain the basic experimental facts about β-decay (the energy spectrum of the emitted electron and other things) that were known at that time. In addition, Fermi's theory is accessible by methods developed in this course. Fermi's theory can be described by a field Hamiltonian,

$$H = H_{0n} + H_{0p} + H_{0e} + H_{0\nu} + H_{\rm int}, \tag{2}$$

where each term with a 0 subscript is the Hamiltonian for a free spin-$\frac{1}{2}$ fermion (neutron, proton, electron, neutrino), each with the general form

$$H_0 = \int d^3\mathbf{x}\; :\psi^\dagger(\mathbf{x})(-i\boldsymbol{\alpha}\cdot\nabla + m\beta)\psi(\mathbf{x}) := \sum_{ps} E(b^\dagger_{ps}b_{ps} + d^\dagger_{ps}d_{ps}), \tag{3}$$

where H_0, ψ, m, E, b_{ps}, d_{ps} etc all implicitly take a subscript $i = n, e, p, \nu$ to indicate the type of particle. The notation is as in class and in the notes. The fermion field ψ has the Fourier expansion,

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{ps}\sqrt{\frac{m}{E}}\Big(b_{ps}u_{ps}e^{i\mathbf{p}\cdot\mathbf{x}} + d^\dagger_{ps}v_{ps}e^{-i\mathbf{p}\cdot\mathbf{x}}\Big), \tag{4}$$

where V is the volume of the box and otherwise the notation is as in class.

As for the interaction Hamiltonian in Eq. (2), Fermi chose it to be the product of four Fermion fields in order to take into account the 4-point vertex in Fig. 1. He also chose a current-current type of interaction, since the electromagnetic interaction is of this type. Fermi's interaction Hamiltonian is

$$H_{\rm int} = g\int d^3\mathbf{x}\; :(\bar\psi_p\gamma^\mu\psi_n)(\bar\psi_e\gamma_\mu\psi_\nu): + \text{h.c.}, \tag{5}$$

where g is a constant, where all fields are evaluated at $\mathbf{x}$ and where h.c. means "Hermitian conjugate." By using this Hamiltonian to calculate the rate of neutron decay and comparing to experiment one can get a value for the constant g (essentially, the Fermi constant).

(a) Explain why the Hermitian conjugate of a term such as $\bar\psi_p\gamma^\mu\psi_n$ is $\bar\psi_n\gamma^\mu\psi_p$ (it is not completely obvious since we have $\bar\psi$ instead of $\psi^\dagger$). This means that the Hermitian conjugate term in the Hamiltonian can be written

$$g\int d^3\mathbf{x}\; :(\bar\psi_\nu\gamma_\mu\psi_e)(\bar\psi_n\gamma^\mu\psi_p): . \tag{6}$$

(b) Show that the interaction Hamiltonian connects the initial and final states in Fig. 1, that is, the matrix element $\langle f|H_{\rm int}|i\rangle$ is nonzero. For this it is sufficient to show that the field part of the matrix element is nonzero. You don't have to write out the matrix element in all detail, you can be schematic, but indicate the important parts for the question at hand. Notice that the interaction gives rise to $2 \times 16 = 32$ possible Feynman diagrams. Show that the Feynman diagram for inverse β-decay,

$$p + e \to n + \nu \tag{7}$$

is one of them. Draw a Feynman diagram connecting the four particles and their antiparticles that is *not* one of the 32.

(c) Inverse β-decay (7) takes place in the final stages of core collapse in a supernova explosion. As the matter is compressed by gravity the top of the electron Fermi sea rises, ultimately reaching relativistic energies ($\sim$ 511 KeV) and beyond. When it reaches $\sim$ 1.3 MeV, the mass difference between the neutron and the proton, the electrons are energetic enough to cause the reaction (7) to take place. Any extra electron energy goes mostly into the energy of the neutrino. As the most energetic electrons are removed, their contribution to the pressure is also eliminated, leading to further gravitational contraction. Simultaneously, protons are converted to neutrons and neutrinos are emitted. In this way a good part of the neutrons in a neutron star are created. The neutrinos from a supernova explosion were actually observed in 1989.

We will use Fermi's theory to compute the cross section for inverse beta decay. Before beginning the quantum mechanics, it's a good idea to practice a little with the conservation laws. Working in the center of mass frame, write down an expression for the total energy E_{tot} of the system as a function of the initial electron 3-momentum $p_e = |\mathbf{p}_e|$. The momentum $\mathbf{p}_e$ is a parameter of the problem and is fixed for the rest of the calculation; therefore so is E_{tot}. Now compute E_{tot} as a function of the final neutrino 3-momentum $p_\nu = |\mathbf{p}_\nu|$. This final 3-momentum $\mathbf{p}_\nu$ should be regarded as a variable, since we will be summing over a collection of final states to get a cross section. Imagine solving for p_ν as a function of E_{tot}. Don't actually do it, since it's a little messy, but call the root obtained p_f (for "final" momentum). In the limit $m_\nu \to 0$, however, it's easy to solve for p_ν as a function of E_{tot}; do this. The whole calculation is done in the center-of-mass frame.

(d) Find an expression for the matrix element $\langle f|H_{\text{int}}|i\rangle$ that is simplified as much as you can make it without explicitly evaluating spinors or spin contractions. Work with box normalization as in class and as reflected by the equations above.

(e) Now write out an expression for the differential cross section $d\sigma/d\Omega$, where $d\Omega$ refers to a cone of small solid angle in some direction, within which the final neutrino momentum lies. Simplify this as much as you can without doing spin sums. You may express things in terms of the momentum p_f found in part (c) and other convenient quantities. Hint: the incident flux can be defined as the number of electrons per unit volume in the initial state times the relative velocities of the initial electron and proton.

(f) Now assume that the incident electron and proton are unpolarized, and that we do not care about the spins of the outgoing neutron and neutrino. Write an expression for the effective differential cross section in this case, and take the limit $m_\nu \to 0$. At the same time you may assume that the proton and neutron are essentially nonrelativistic, so $E \approx m$ and $v \ll 1$ for these particles. This is appropriate for the astrophysical application discussed above. Do the spin sums to get a practical formula for the effective $d\sigma/d\Omega$.

(g) Integrate this to get a total cross-section.

Of course the answer will depend on the constant g. By doing a separate calculation, the neutron lifetime can be expressed in terms of g. Since the neutron lifetime is known (about 15 minutes), one can get a numerical value of g, and hence real numbers that one can use in the cross section for inverse beta decay. Such considerations are obviously important in understanding supernova explosions.