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Solutions to the Dirac Equation for a Free Particle

3.1 Plane-wave Solutions

We have seen that the Dirac theory meets the requirements of Lorentz covariance and that the positive-energy solutions to the Dirac equation have a sensible nonrelativistic correspondence.

Further insight into the nature and interpretation of solutions of the Dirac equation may be gained by considering the free-particle equation. The four solutions corresponding to a free particle at rest were given in (1.24) and are written in the combined form

$$\psi^r(x) = w^r(0)e^{-(i\epsilon_r mc^2/\hbar)t} \qquad r = 1, 2, 3, 4 \tag{3.1}$$

with

$$\epsilon_r = \begin{cases} +1 & r = 1, 2 \\ -1 & r = 3, 4 \end{cases}$$

The spinors are

$$w^1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad w^2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad w^3(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad w^4(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.2}$$

in this representation, Eq. (1.17), of the Dirac matrices. The first two solutions describe the two spin degrees of freedom of a Schrödinger-Pauli electron. The "negative-energy" solutions, $r = 3$ and 4, remain to be interpreted. They are all eigenfunctions of $\sigma_z = \sigma_{12}$ with eigenvalues $+1$ and -1. The Lorentz transformation (2.10) may be used to build the free-particle solutions for an arbitrary velocity. By transforming to a coordinate system moving with velocity $-\mathbf{v}$ relative to that of the solutions at rest, we construct free-particle wave functions for an electron with the observed velocity $+\mathbf{v}$.

In order to exhibit the general space-time coordinate variation, we need only express the exponent in (3.1) in invariant form:

$$\exp\left(-i\epsilon_r \frac{mc^2}{\hbar}\, t\right) = \exp\left(-i\epsilon_r \frac{p_\mu{}^{(0)}x^\mu}{\hbar}\right) = \exp\left(-i\epsilon_r \frac{p_\mu x^{\mu'}}{\hbar}\right) \tag{3.3}$$

where $x^{\mu'} = a^\mu{}_\nu x^\nu$ and $p^\mu = a^\mu{}_\nu p^{\nu(0)} = a^\mu{}_0 mc$; our notation throughout is such that $p^0 \equiv E/c = +\sqrt{\mathbf{p}^2 + m^2c^2} > 0$. The positive- and negative-energy solutions transform among themselves separately and do not mix with each other under proper Lorentz transformations, as well as under spatial inversions. This is seen to follow from (3.3), since the four-momentum of a free particle is time-like, $p^\mu p_\mu = m^2c^2 > 0$. Therefore, p_μ is within the light cones in p space. Under the transformations mentioned above, the future and past light cones, and hence the positive- and negative-energy solutions, remain distinct.

We transform the spinors with

$$S = e^{-(i/2)\omega\sigma_{01}} \tag{3.4}$$

according to (2.23), where for simplicity we have specified the velocity to lie along the x axis. The Lorentz angle ω in (3.4) is given by $\omega = \tanh^{-1}(-v/c) = -\tanh^{-1}(v/c)$ and differs by a minus sign from (2.21), since we are transforming to a system moving in the x direction with velocity $-\mathbf{v}$.

Applying the transformation (3.4) to the spinors (3.2), we find

$$w^r(\mathbf{p}) = e^{-(i\omega/2)\sigma_{01}}w^r(0) = \left(\cosh\frac{\omega}{2} - \alpha_1 \sinh\frac{\omega}{2}\right) w^r(0)$$

$$= \cosh\frac{\omega}{2}\begin{bmatrix} 1 & 0 & 0 & -\tanh\frac{\omega}{2} \\ 0 & 1 & -\tanh\frac{\omega}{2} & 0 \\ 0 & -\tanh\frac{\omega}{2} & 1 & 0 \\ -\tanh\frac{\omega}{2} & 0 & 0 & 1 \end{bmatrix} w^r(0) \tag{3.5}$$

From the form (3.2) for $w^r(0)$, it is clear that the rth column of this transformation matrix is identically the column spinor corresponding to $w^r(\mathbf{p})$. We may reexpress it in terms of the energy and momentum of the particle by using the trigonometric identities,

$$-\tanh\frac{\omega}{2} = \frac{-\tanh\omega}{1+\sqrt{1-\tanh^2\omega}} = \frac{v/c}{1+\sqrt{1-(v^2/c^2)}} = \frac{pc}{E+mc^2}$$

and
$$\cosh\frac{\omega}{2} = \sqrt{\frac{E+mc^2}{2mc^2}} \tag{3.6}$$

Also, we may generalize (3.5) to the case of arbitrary direction of the velocity $\mathbf{v}$. In this case the matrix I in (2.19) is replaced by

$$I^\mu{}_\nu = \begin{bmatrix} 0 & -\cos\alpha & -\cos\beta & -\cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{bmatrix}$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of the velocity

v, and in the transformation matrix

$$\sigma_{\mu\nu} I_n^{\mu\nu} = 2(\sigma_{01} \cos\alpha + \sigma_{02} \cos\beta + \sigma_{03} \cos\gamma) = -2i \frac{\boldsymbol{\alpha} \cdot \mathbf{v}}{|\mathbf{v}|}$$

This gives, with the aid of (3.6),

$$S = \exp\left(-\frac{\omega}{2} \frac{\boldsymbol{\alpha} \cdot \mathbf{v}}{|\mathbf{v}|}\right)$$

$$= \sqrt{\frac{E + mc^2}{2mc^2}} \begin{bmatrix} 1 & 0 & \dfrac{p_z c}{E + mc^2} & \dfrac{p_- c}{E + mc^2} \\ 0 & 1 & \dfrac{p_+ c}{E + mc^2} & \dfrac{-p_z c}{E + mc^2} \\ \dfrac{p_z c}{E + mc^2} & \dfrac{p_- c}{E + mc^2} & 1 & 0 \\ \dfrac{p_+ c}{E + mc^2} & \dfrac{-p_z c}{E + mc^2} & 0 & 1 \end{bmatrix} \tag{3.7}$$

where $p_\pm \equiv p_x \pm ip_y$. The general form of a free-particle solution is

$$\psi^r(x) = w^r(\mathbf{p}) e^{-i\epsilon_r (p_\mu x^\mu / \hbar)} \tag{3.8}$$

where the rth column of (3.7) gives the corresponding spinor $w^r(\mathbf{p})$ in the representation of the γ matrices given by Eq. (1.17).

The $w^r(\mathbf{p})$ satisfy the following useful relations:

$$(\not{p} - \epsilon_r mc) w^r(\mathbf{p}) = 0 \qquad \bar{w}_r(\mathbf{p})(\not{p} - \epsilon_r mc) = 0 \tag{3.9a}$$

$$\bar{w}^r(\mathbf{p}) w^{r'}(\mathbf{p}) = \delta_{rr'} \epsilon_r \tag{3.9b}$$

$$\sum_{r=1}^{4} \epsilon_r w_\alpha^r(\mathbf{p}) \bar{w}_\beta^r(\mathbf{p}) = \delta_{\alpha\beta} \tag{3.9c}$$

Equation (3.9a), obtained by applying the Dirac operator $(i\nabla\!\!\!/ - m)$ to (3.8), states the Dirac equation for a free particle in momentum space. For $r = 1$ or 2, $\epsilon_r = +1$ and $(\not{p} - mc)w^r(\mathbf{p}) = 0$. This is the equation for the two positive-energy solutions given by the first two columns of (3.7). In this representation their third and fourth components are the "small components" in a nonrelativistic approximation, and they reduce to Eqs. (1.29) and (1.31) in the absence of external fields. For the negative-energy solutions the "large" and "small" components are interchanged in (3.7). We also introduce the adjoint spinor according to the definition in (2.28): $\bar{w}^r(\mathbf{p}) \equiv w^{r\dagger}(\mathbf{p})\gamma_0$. It satisfies the adjoint wave equation

$$\bar{w}^r(\mathbf{p})(\not{p} - \epsilon_r mc) = 0 \tag{3.10}$$

which is obtained by taking the hermitian conjugate of (3.9*a*) and multiplying from the right by γ^0 with the aid of the identities $(\gamma^0)^2 = +1$ and $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$.

Equation (3.9*b*) is a covariant normalization statement. The bilinear form $\bar{w}^r(\mathbf{p})w^{r'}(\mathbf{p})$ is a Lorentz scalar as discussed in the preceding chapter [see Eq. (2.38)], and so we verify (3.9*b*) simply by returning to the rest solutions (3.2). The probability density $w^{r\dagger}(\mathbf{p})w^r(\mathbf{p})$ will not be an invariant but transforms as the fourth component of a vector according to (2.27). Calculating from the columns of (3.7) we find

$$w^{r\dagger}(\epsilon_r\mathbf{p})w^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E}{mc^2}\,\delta_{rr'} \tag{3.11}$$

This shows that the probability density acquires the correct factor E/mc^2 to compensate the Lorentz contraction of the volume element along the direction of motion and to preserve thereby the normalization of the invariant probability. Notice that (3.9*b*) is an orthogonality statement between a spinor and its adjoint of the same momentum $\mathbf{p}$, whereas in (3.11) the positive-energy spinor is orthogonal to its hermitian conjugate spinor of negative energy and reversed momentum. Thus two plane-wave solutions of the same spatial momentum $\mathbf{p}$ but of opposite energy are orthogonal in the sense that $\psi^{r\dagger}(x)\psi^{r'}(x) = 0$ if $r = 1, 2$ and $r' = 3, 4$, or vice versa.

Equation (3.9*c*) is a completeness statement applying to the four Dirac spinors for a given momentum. It is clearly true for a free particle at rest. To prove it for an arbitrary momentum, we can make an appropriate Lorentz transformation to the rest system and then use (3.2) to find

$$\sum_{r=1}^{4} \epsilon_r w_\alpha^r(\mathbf{p})\bar{w}_\beta^r(\mathbf{p}) = \sum_{r=1}^{4} \epsilon_r S_{\alpha\gamma}\left(-\frac{\mathbf{p}}{E}\right) w_\gamma^r(0)\bar{w}_\lambda^r(0) S_{\lambda\beta}^{-1}\left(-\frac{\mathbf{p}}{E}\right)$$
$$= S_{\alpha\gamma}\delta_{\gamma\lambda}S_{\lambda\beta}^{-1} = \delta_{\alpha\beta}$$

That $\bar{w}$ and not $w^\dagger$ appears in the completeness relation is due to the relation $S^\dagger = \gamma^0 S^{-1}\gamma^0$ derived in (2.26) and again reflects the fact that the Lorentz transformation is not unitary.

By using the rotation operators

$$S = e^{(i/2)\varphi\boldsymbol{\sigma}\cdot\mathbf{s}}$$

upon the solutions (3.2) for the electron *at rest* and polarized in the z direction, it is possible to form states which are polarized in any arbitrary direction $\mathbf{s}$. In particular, the defining relation for such states is

$$\boldsymbol{\sigma} \cdot \mathbf{s} w = w$$

if the spinor w corresponds to a particle polarized along direction of the unit vector $\mathbf{s}$. The specific form of these solutions is similar to that of the two-component Pauli theory owing to the structure of $\boldsymbol{\sigma}$ in (2.24).

In this description it is convenient to introduce a different notation. Let $u(p,s)$ denote the spinor which is a positive-energy solution of the Dirac equation with momentum p^μ and spin s^μ. Thus $u(p,s)$ satisfies the equation

$$(\not{p} - mc)_{\alpha\beta} u_\beta(p,s) = 0 \tag{3.12}$$

The spin vector s^μ is defined in terms of the polarization vector $\check{\mathbf{s}}$ in the rest frame by $s^\mu = a^\mu{}_\nu \check{s}^\nu$, where $\check{s}^\nu = (0,\check{\mathbf{s}})$ and the $a^\mu{}_\nu$ are the transformation coefficients to the rest frame, that is, $p^\mu = a^\mu{}_\nu \check{p}^\nu$, where $\check{p}^\nu = (m,\mathbf{0})$. Notice that $s_\mu s^\mu = -1$ and that $\check{p}^\mu \check{s}_\mu = 0$ and therefore $p^\mu s_\mu = 0$. In the rest frame u satisfies

$$\boldsymbol{\sigma} \cdot \check{\mathbf{s}} u(\check{p},\check{s}) = u(\check{p},\check{s}) \tag{3.13}$$

Similarly let $v(p,s)$ denote a negative-energy solution

$$(\not{p} + mc) v(p,s) = 0 \tag{3.14}$$

with polarization $-\check{\mathbf{s}}$ in the rest frame, that is,

$$\boldsymbol{\sigma} \cdot \check{\mathbf{s}} v(\check{p},\check{s}) = -v(\check{p},\check{s}) \tag{3.15}$$

The $u(p,s)$ and $v(p,s)$ are related to the $w^r(\mathbf{p})$ by

$$\begin{aligned} w^1(\mathbf{p}) &= u(p,u_z) \\ w^2(\mathbf{p}) &= u(p,-u_z) \\ w^3(\mathbf{p}) &= v(p,-u_z) \\ w^4(\mathbf{p}) &= v(p,u_z) \end{aligned} \tag{3.16}$$

with u_z^μ a four-vector, which in the rest frame takes the form

$$\check{u}_z^\mu = (0,\check{\mathbf{u}}_z) = (0,0,0,1)$$

An arbitrary spinor is thus specified by the momentum p_μ, the sign of the energy, and the polarization in the rest frame $\check{s}_\mu$.

3.2 Projection Operators for Energy and Spin

In practical calculations, it is often convenient to have operators which project out a spinor of given sign of energy and polarization.

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These projection operators are the four-dimensional analogues of the nonrelativistic two-component operators

$$P_{\pm} = \frac{1 \pm \sigma_z}{2}$$

which project out of an arbitrary state the spin-up or spin-down amplitude.

For the Dirac equation, we search for four operators which project from a given plane-wave solution of momentum $\mathbf{p}$ the four independent solutions corresponding to positive and negative energy and to spin up and spin down along a given direction. We would like these operators in a covariant form so that we may transform with ease among different Lorentz systems, as will prove useful in practical calculations.

The four projection operators are denoted by $P_r(\mathbf{p}) \equiv P(p_\mu, u_z, \epsilon)$ and are defined to satisfy the following properties:

$$P_r(\mathbf{p})w^{r'}(\mathbf{p}) = \delta_{rr'}w^{r'}(\mathbf{p})$$

or equivalently

$$P_r(\mathbf{p})P_{r'}(\mathbf{p}) = \delta_{rr'}P_r(\mathbf{p}) \tag{3.17}$$

An operator which projects out positive- or negative-energy eigenstates for a given $\mathbf{p}$ may be found directly from (3.9*a*), already in covariant form. We denote it by

$$\Lambda_r(p) = \frac{\epsilon_r \not{p} + mc}{2mc}$$

or, alternatively,

$$\Lambda_{\pm}(p) = \frac{\pm \not{p} + mc}{2mc} \tag{3.18}$$

By direct calculation, using $\not{p}\not{p} = p^2 = m^2c^2$, we verify that

$$\Lambda_r(p)\Lambda_{r'}(p) = \frac{m^2c^2(1 + \epsilon_r\epsilon_{r'}) + mc\not{p}(\epsilon_r + \epsilon_{r'})}{4m^2c^2} = \left(\frac{1 + \epsilon_r\epsilon_{r'}}{2}\right)\Lambda_r(p)$$

that is,

$$\Lambda_+^2(p) = \Lambda_+(p)$$

$$\Lambda_+(p)\Lambda_-(p) = 0$$

Also notice that

$$\Lambda_+(p) + \Lambda_-(p) = 1$$

To exhibit the analogous operator for the spin, we go to the rest frame, where the spin is most easily described, and try to find a projection operator which may be cast into covariant form. The natural candidate for a spin-up particle is $(1 + \sigma_z)/2$. In the same way as the two-component nonrelativistic spin projection operator is liberated from explicit dependence upon the z direction by rewriting $(1 + \sigma_z)/2$ as a scalar,

$$\frac{1 + \boldsymbol{\sigma} \cdot \check{\mathbf{u}}_z}{2}$$

we try to write the Dirac spin projection operator in scalar form by using the four-vector $\check{u}_z^\mu$, that is

$$\frac{1 + \sigma_z}{2} = \frac{1 + \gamma_5\gamma_3\check{u}_z^3\gamma_0}{2} = \frac{1 + \gamma_5\check{\not{u}}_z\gamma_0}{2}$$

This may now be cast into covariant form by eliminating the γ_0. Because we are in the rest frame, γ_0 acting upon the Dirac spinor becomes ± 1. With the conventions established in (3.14) and (3.15), the covariant Dirac spin projection operator is finally

$$\Sigma(u_z) = \frac{1 + \gamma_5\not{u}_z}{2}$$

or for a general spin vector s^μ, with $s^\mu p_\mu = 0$,

$$\Sigma(s) = \frac{1 + \gamma_5\not{s}}{2} \tag{3.19}$$

Thus in the rest frame

$$\Sigma(\check{u}_z)w^1(0) = \frac{1 + \gamma_5\check{\not{u}}_z}{2}\,w^1(0) = \frac{1 + \sigma_z}{2}\,w^1(0) = w^1(0) \tag{3.20}$$

and

$$\Sigma(-\check{u}_z)w^2(0) = w^2(0)$$

Similarly, for the negative-energy spinors

$$\Sigma(-\check{u}_z)w^3(0) = \frac{1 - \gamma_5\check{\not{u}}_z}{2}\,w^3(0) = \frac{1 + \gamma_5\check{\not{u}}_z\gamma_0}{2}\,w^3(0)$$

$$= \frac{1 + \sigma_z}{2}\,w^3(0) = w^3(0) \tag{3.21}$$

and

$$\Sigma(\check{u}_z)w^4(0) = w^4(0)$$

In terms of the definitions (3.16) of the spinors u and v, these are

$$\Sigma(u_z)u(p,u_z) = u(p,u_z)$$
$$\Sigma(u_z)v(p,u_z) = v(p,u_z)$$
$$\Sigma(-u_z)u(p,u_z) = \Sigma(-u_z)v(p,u_z) = 0$$

Because of the covariant form of the projection operator Σ, we may write for any polarization vector $s^\mu (s^\mu p_\mu = 0)$ that

$$\begin{aligned} \Sigma(s)u(p,s) &= u(p,s) \\ \Sigma(s)v(p,s) &= v(p,s) \\ \Sigma(-s)u(p,s) &= \Sigma(-s)v(p,s) = 0 \end{aligned} \tag{3.22}$$

With the four projection operators $\Lambda_\pm(p)$ and $\Sigma(\pm s)$ we can now completely specify free-particle motion in terms of four-momentum p_μ, sign of energy ϵ, and polarization s^μ with $s^\mu p_\mu = 0$. In particular, we construct from (3.18) and (3.19) the four projection operators

$$\begin{aligned} P_1(\mathbf{p}) &= \Lambda_+(p)\Sigma(u_z) \\ P_2(\mathbf{p}) &= \Lambda_+(p)\Sigma(-u_z) \\ P_3(\mathbf{p}) &= \Lambda_-(p)\Sigma(-u_z) \\ P_4(\mathbf{p}) &= \Lambda_-(p)\Sigma(u_z) \end{aligned}$$

Notice that $[\Sigma(s), \Lambda_\pm(p)] = 0$ for all vectors satisfying $s^\mu p_\mu = 0$, since $\not p$ anticommutes with both γ_5 and $\not s$. From this it follows that these $P_r(\mathbf{p})$ satisfy the defining relations (3.17).

We shall rely upon these projection operators very frequently in developing rapid and efficient calculational techniques. They permit us to use closure methods, thus avoiding the necessity of writing out matrices and spinor solutions component by component.

In order to achieve an invariant formulation, we have introduced negative-energy solutions of momentum $\mathbf{p}$ which, according to (3.8), are eigenfunctions of the momentum operator p with eigenvalue $-\mathbf{p}$. Similarly, according to (3.19) and (3.21), the negative-energy solutions representing spin-up and spin-down states reduce in their rest frames to eigenfunctions of σ_z with eigenvalues -1 and $+1$, respectively. The physical motivation for this apparently backward association of eigenvalues for the negative-energy solutions will appear when we come to the hole theory in Chap. 5.

3.3 Physical Interpretation of Free-particle Solutions and Packets

We may now superpose the plane-wave solutions at our disposal to construct localized packets. These packets are still solutions of the free Dirac equation, as required by the superposition principle, since the Dirac equation is linear. We study them to gain further insight into the interpretation of the free-particle solutions.

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To begin, we form a packet by superposing positive-energy solutions only:

$$\psi^{(+)}(\mathbf{x},t) = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{mc^2}{E}} \sum_{\pm s} b(p,s)u(p,s)e^{-ip_\mu x^\mu/\hbar} \tag{3.23}$$

To normalize the expansion coefficients $b(p,s)$ to unit probability, we call on the spinor orthogonality relations (3.11) and find[1]

$$\begin{aligned}\int \psi^{(+)\dagger}(\mathbf{x},t)\psi^{(+)}(\mathbf{x},t)\, d^3x &= \int d^3p\, \frac{mc^2}{E} \sum_{\pm s, \pm s'} b^*(p,s')b(p,s)u^\dagger(p,s')u(p,s) \\ &= \int d^3p \sum_{\pm s} |b(p,s)|^2 = 1\end{aligned} \tag{3.24}$$

The average current for such a packet is given by the expectation value of the velocity operator

$$\mathbf{J}^{(+)} = \int \psi^{(+)\dagger} c\boldsymbol{\alpha}\psi^{(+)}\, d^3x \tag{3.25}$$

In evaluating this we use the following important relation between the three four-vectors that can be formed from free-particle solutions:

For $\psi_1(x)$ and $\psi_2(x)$ any two solutions to the Dirac equation, $(\not p - mc)\psi(x) = 0$,

$$c\bar\psi_2\gamma^\mu\psi_1 = \frac{1}{2m}\left[\bar\psi_2 \mathrm{p}^\mu\psi_1 - (\mathrm{p}^\mu\bar\psi_2)\psi_1\right] - \frac{i}{2m}\,\mathrm{p}_\nu(\bar\psi_2\sigma^{\mu\nu}\psi_1) \tag{3.26}$$

To prove (3.26), we observe that if a^μ and b^μ are two arbitrary four-vectors

$$\begin{aligned}\not a \not b &= a_\mu b_\nu[\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)] \\ &= a^\mu b_\mu - ia^\mu b^\nu\sigma_{\mu\nu}\end{aligned} \tag{3.27}$$

[1] We collect here familiar properties of the Dirac δ function used in deriving (3.24):

$$\int_{-\infty}^{\infty} dx\, e^{i(s-a)x} = 2\pi\delta(s-a)$$

$$\int_{\left[\substack{\text{interval}\\ \text{including } s=a}\right]} ds\, \delta(s-a)f(s) = f(a)$$

if $f(s)$ has no singularities in the interval of integration;

$$\delta\left(\frac{s}{c}\right) = |c|\delta(s) \qquad |c| \neq 0$$

The δ function is mathematically respectable in the sense of distribution theory; see, for instance, M. J. Lighthill, "Introduction to Fourier Analysis and Generalized Functions," Cambridge University Press, London, 1958.

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Then with the Dirac equation we construct

$$0 = \bar{\psi}_2(-\overleftarrow{\not{p}} - mc)\not{a}\psi_1 + \bar{\psi}_2\not{a}(\overrightarrow{\not{p}} - mc)\psi_1$$
$$= -2mc\bar{\psi}_2\not{a}\psi_1 + \bar{\psi}_2[a^\mu \overrightarrow{p}_\mu - ia^\mu \overrightarrow{p}^\nu \sigma_{\mu\nu} - \overleftarrow{p}^\mu a_\mu + i\overleftarrow{p}^\mu a^\nu \sigma_{\mu\nu}]\psi_1$$
$$(\bar{\psi}\overleftarrow{\not{p}} = p_\mu \bar{\psi}\gamma^\mu)$$

and (3.26) emerges as the coefficient of an arbitrary vector a^μ.

This identity is known as the Gordon decomposition.[1] It expresses the Dirac current as the sum of a convection current similar to the nonrelativistic one, and a spin current.

With the help of (3.26) for the special case $\psi_2 = \psi_1 = \psi$ and (3.23), we now find for the current (3.25)

$$J_i^{(+)} = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi\hbar)^3} \frac{mc^2}{\sqrt{EE'}} \sum_{\pm s, \pm s'} b^*(p',s')b(p,s)e^{i(p'-p)^\mu x_\mu/\hbar}$$
$$\times \frac{1}{2m}\bar{u}(p',s')[(p'_i + p_i) + i\sigma_i{}^\nu(p'_\nu - p_\nu)]u(p,s)$$
$$= \int d^3p \frac{p_i c^2}{E} \sum_{\pm s} |b(p,s)|^2 \tag{3.28}$$

According to the normalization (3.24), the current can be written

$$\mathbf{J}^{(+)} = \langle c\boldsymbol{\alpha}\rangle_+ = \left\langle \frac{c^2\mathbf{p}}{E} \right\rangle_+ = \langle \mathbf{v}_{gp}\rangle_+ \tag{3.29}$$

where $\langle\ \ \rangle_+$ denotes expectation value with respect to a positive-energy packet. Thus the average current for an arbitrary packet formed of positive-energy solutions is just the classical group velocity. The corresponding statement is familiar in the nonrelativistic Schrödinger theory.

Now we come to an important difference in the relativistic theory. In the Schrödinger theory the velocity operator appearing in the current is just $\mathbf{p}/m$ and is a constant of the motion for free particles. The current is not, however, proportional to the momentum in the Dirac theory, and whereas the Ehrenfest relation (1.27) has shown that $\dfrac{d}{dt}\mathbf{p} = 0$ for free-particle motion, the velocity operator $c\boldsymbol{\alpha}$ is not constant, since $[\boldsymbol{\alpha},H] \neq 0$. Indeed in constructing eigenfunctions of $c\boldsymbol{\alpha}$ we have to include both positive- and negative-energy solutions, since the eigenvalues of $c\alpha^i$ are $\pm c$ whereas $|\langle c\alpha^i\rangle_+| < c$, according to (3.29).

[1] W. Gordon, *Z. Physik*, **50**, 630 (1928).

Let us now enlarge our considerations to include the negative- as well as positive-energy solutions in forming a packet from the complete set of free-particle solutions. We generalize (3.23) to

$$\psi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{mc^2}{E}} \sum_{\pm s} [b(p,s)u(p,s)e^{-ip^\mu x_\mu/\hbar} + d^*(p,s)v(p,s)e^{+ip^\mu x_\mu/\hbar}] \quad (3.30)$$

again normalized to unit probability. A short calculation gives for the probability

$$\int d^3x\, \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t) = \int d^3p \sum_{\pm s} [|b(p,s)|^2 + |d(p,s)|^2] = 1$$

and for the current for such a packet[1]

$$\begin{aligned} J^k = \int d^3p \Bigg\{ &\sum_{\pm s} [|b(p,s)|^2 + |d(p,s)|^2] \frac{p^k c^2}{E} \\ &+ ic \sum_{\pm s, \pm s'} b^*(-p,s')d^*(p,s)e^{2ix_0p_0/\hbar}\bar{u}(-p,s')\sigma^{k0}v(p,s) \\ &- ic \sum_{\pm s, \pm s'} b(-p,s')d(p,s)e^{-2ix_0p_0/\hbar}\bar{v}(p,s')\sigma^{k0}u(-p,s) \Bigg\} \end{aligned} \quad (3.31)$$

In addition to the time-independent group velocity there now appear cross terms between the positive- and negative-energy solutions which oscillate rapidly in time with frequencies

$$\frac{2p_0c}{\hbar} > \frac{2mc^2}{\hbar} = 2 \times 10^{21}\ \text{sec}^{-1}$$

This rapid oscillation, or *zitterbewegung*,[2] is proportional to the amplitude of the negative-energy solutions in the packet. We have as yet no physical interpretation of these solutions, but we may ask when to expect them to be present in the packet with appreciable amplitude. The general form of a free-particle solution (3.30) shows explicitly by the time independence of $b(p,s)$ that a packet initially formed with positive-energy solutions only does not develop negative-energy components in the absence of forces. However, a packet formed to represent an electron somehow localized initially in a region

[1] Despite a certain inconsistency, we denote hereafter

$$u(\sqrt{\mathbf{p}^2 + m^2}, -\mathbf{p}, s) \equiv u(-p,s)$$

with similar conventions for expansion coefficients b, d^*, etc.

[2] E. Schrödinger, *Sitzber. Preuss. Akad. Wiss. Physik-Math.*, **24**, 418 (1930).

of finite extent generally includes solutions of both signs of energy. Consider, for example, the solution

$$\psi(\mathbf{r},0,s) = (\pi d^2)^{-3/4} e^{-\frac{1}{2}r^2/d^2} w^1(0) \tag{3.32}$$

which corresponds to a Gaussian density distribution of half-width $\sim d$ about the origin at time $t = 0$. At a later time t it can be expressed as a packet (3.30) with the coefficients b and d^* fixed by the initial conditions, viz., at $t = 0$

$$\int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{mc^2}{E}} \sum_{\pm s} [b(p,s)u(p,s)e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} + d^*(p,s)v(p,s)e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}]$$
$$= (\pi d^2)^{-3/4} e^{-\frac{1}{2}r^2/d^2} w^1(0)$$

Taking the Fourier transform and using

$$\int_{-\infty}^{\infty} d^3r\, e^{-r^2/2d^2} e^{i\mathbf{l}\cdot\mathbf{r}/\hbar} = (2\pi d^2)^{3/2} e^{-\frac{1}{2}l^2d^2/\hbar^2}$$

we find

$$\sqrt{\frac{mc^2}{E}} \sum_{\pm s'} [b(p,s')u(p,s') + d^*(-p,s')v(-p,s')] = \left(\frac{d^2}{\pi\hbar^2}\right)^{3/4} e^{-\frac{1}{2}p^2d^2/\hbar^2} w^1(0)$$

The orthogonality relation (3.11) gives

$$\begin{aligned} b(p,s) &= \sqrt{\frac{mc^2}{E}} \left(\frac{d^2}{\pi\hbar^2}\right)^{3/4} e^{-p^2d^2/2\hbar^2} u^\dagger(p,s) w^1(0) \\ d^*(-p,s) &= \sqrt{\frac{mc^2}{E}} \left(\frac{d^2}{\pi\hbar^2}\right)^{3/4} e^{-p^2d^2/2\hbar^2} v^\dagger(-p,s) w^1(0) \end{aligned} \tag{3.33}$$

Thus the amplitude d^* of the negative-energy solutions in the packet (3.32) is nonzero. Relative to the positive-energy components b it is reduced by the ratio of the upper, or small, components of v to the upper, or large, components of u, that is, by $\sim pc/(E + mc^2)$. This shows that the negative-energy amplitudes are appreciable for momenta $\sim mc$. We also see in (3.33), however, that the packet is composed predominantly of momenta $p \lesssim \hbar/d$. Therefore, this packet must be localized in a region of space comparable with the electron Compton wavelength, that is, with $d \sim \hbar/mc$, before the negative-energy solutions enter appreciably.[1]

[1] For a discussion of the position coordinate of a positive-energy Dirac electron see T. D. Newton and E. P. Wigner, *Rev. Mod. Phys.*, **21**, 400 (1949).

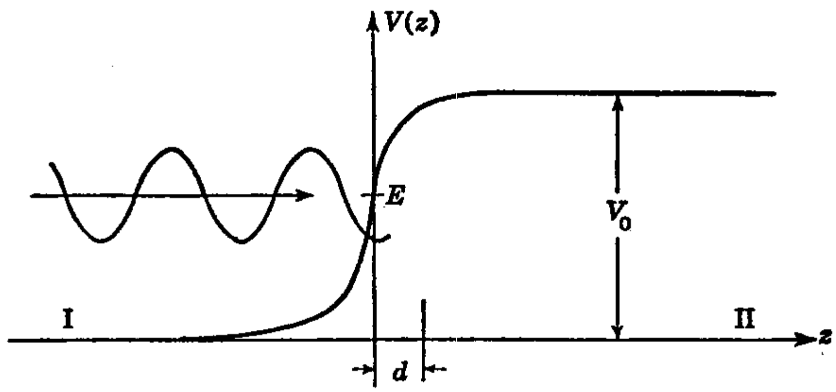

Fig. 3-1 Potential barrier confining electron of energy E in region I to the left.

This result can be equally well inferred on dimensional groups using $\Delta p\, \Delta x \sim \hbar$ without reference to the particular gaussian shape. In discussing problems and interactions in which the electron is "spread out" over distances large compared with its Compton wavelength, we may simply ignore the existence of the uninterpreted negative-energy solutions and hope to obtain physically sensible and accurate results. This will not work, however, in situations which find electrons localized to distances comparable with $\hbar/mc$. The negative-frequency amplitudes will then be appreciable, the zitterbewegung terms in the current important, and indeed we shall find ourselves beset by paradoxes and dilemmas which defy interpretation within the framework so far developed by the Dirac theory of an electron. A celebrated example of these difficulties is the Klein paradox,[1] illustrated by the following example.

In order to localize electrons, we must introduce strong external forces confining them to the desired region. Suppose, for example, we want to confine a free electron of energy E to region I to the left of the origin $z = 0$ in the one-dimensional potential diagram of Fig. 3.1. If the electron is not to be found more than a distance d to the right of $z = 0$, in region II, then V must rise sharply within an interval $z \lesssim d$ to a height $V_0 > E$ so that the solution in II falls off with a characteristic width $\lesssim d$. This is as in the Schrödinger theory, until the confining length d shrinks to $\sim\hbar/mc$ and $V_0 - E$ increases beyond mc^2. To see what happens, let us consider an electrostatic potential with a sharp boundary as in Fig. 3.2 and calculate the reflected and transmitted current for an electron of wave number k incident from the left with spin up along the z direction. The positive-energy solutions for the incident and reflected waves in region I may be

[1] O. Klein, *Z. Physik*, **53**, 157 (1929).

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written

$$\psi_{\text{inc}} = ae^{ik_1z}\begin{bmatrix} 1 \\ 0 \\ \dfrac{ck_1\hbar}{E+mc^2} \\ 0 \end{bmatrix} \tag{3.34}$$

$$\psi_{\text{ref}} = be^{-ik_1z}\begin{bmatrix} 1 \\ 0 \\ -\dfrac{ck_1\hbar}{E+mc^2} \\ 0 \end{bmatrix} + b'e^{-ik_1z}\begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{ck_1\hbar}{E+mc^2} \end{bmatrix}$$

For the transmitted wave we need the solutions of the Dirac equation in the presence of a constant external potential $e\Phi = V_0$. These differ from the free-particle solutions only by the substitution $p_0 = (1/c)(E - V_0)$, so that in region II

$$\hbar^2 k_2{}^2 c^2 = (E - V_0)^2 - m^2c^4 = (E - mc^2 - V_0)(E + mc^2 - V_0)$$

We therefore write the transmitted wave of positive energy $E > 0$ as

$$\psi_{\text{trans}} = de^{ik_2z}\begin{bmatrix} 1 \\ 0 \\ \dfrac{c\hbar k_2}{E - V_0 + mc^2} \\ 0 \end{bmatrix} + d'e^{ik_2z}\begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-c\hbar k_2}{E - V_0 + mc^2} \end{bmatrix} \tag{3.35}$$

The amplitudes d and d' are fixed by continuity of the solution at

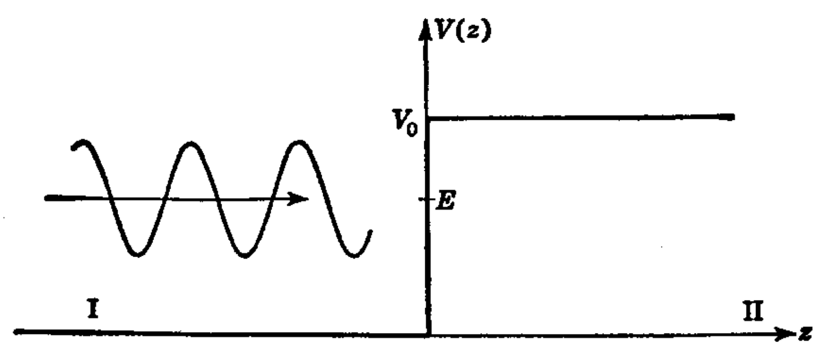

Fig. 3-2 Electrostatic potential idealized with a sharp boundary, with an incident free electron wave of energy E moving to the right in region I. For $V_0 > E + mc^2$ the reflected current from the potential exceeds the incident one; this is an example of the Klein paradox.

the potential boundary as required by current conservation:

$$\begin{aligned} a + b &= d \\ a - b &= \frac{k_2}{k_1}\frac{E + mc^2}{E - V_0 + mc^2}\,d \equiv rd \qquad (3.36) \\ b' &= d' = 0 \qquad \text{(there is no spin flip)} \end{aligned}$$

If $V_0 > 0$ and $|E - V_0| < mc^2$, the wave number is imaginary, $k_2 = +i|k_2|$, and the solution in region II is a decaying exponential corresponding to damping in a distance $d > \hbar/mc$. However, as we increase the height of the barrier beyond $V_0 = E + mc^2$ in order to further confine the electron, the transmitted wave becomes oscillatory. The transmitted and reflected currents may be computed, and we find

$$\frac{j_{\text{trans}}}{j_{\text{inc}}} = \frac{4r}{(1 + r)^2} \qquad \frac{j_{\text{ref}}}{j_{\text{inc}}} = \frac{(1 - r)^2}{(1 + r)^2} = 1 - \frac{j_{\text{trans}}}{j_{\text{inc}}} \qquad (3.37)$$

Whereas the form of these results reminds us of the analogous predictions of the Schrödinger theory, we must now observe that, by (3.36) and the above condition $V_0 > E + mc^2$, $r < 0$. So we find in (3.37) a result contradicting our ordinary reasoning by indicating a *negative* transmitted current and a reflected current *exceeding* the incident one. What is the source of a current in region II moving left in Fig. 3.2 into region I in this case of $V_0 > E + mc^2$? We increased the potential height V_0 beyond $E + mc^2$ in attempting to localize the solution within one Compton wavelength $\hbar/mc$, but ended up with undamped oscillatory solutions instead. How do we understand this? Only by understanding and interpreting the negative-energy solutions. It is clear from the packet discussion that they enter prominently in solutions localized within $\hbar/mc$. It is equally clear from the above calculation of the currents that our physical picture of what is going on also fails at these distances.

We shall tackle and resolve these questions starting in Chap. 5. Before doing this let us look in the vast, if limited, domain of physical problems where the applied forces are weak and smoothly varying on a scale whose energy unit is mc^2 and whose distance unit is $\hbar/mc$. Here we may expect to find fertile fields for application of the Dirac equation and theory for positive-energy electrons.

Problems

1. Derive (3.11) in a representation-free way directly from the Dirac equation.

2. Prove that (3.9*c*) is independent of the specific representation of the Dirac spinors.

3. Derive (3.31) for the current in a general packet (3.30).

4. Verify (3.36) as the conditions for current conservation.

5. Find the energy levels of a Dirac particle in a one-dimensional box of depth V_0 and width a.

6. Verify the completeness relation

$$\sum_{r=1}^{4} w_\alpha{}^r(\epsilon^r \mathbf{p}) w_\beta{}^{r\dagger}(\epsilon^r \mathbf{p}) = \frac{E}{m}\,\delta_{\alpha\beta}$$