2

Lorentz Covariance of the Dirac Equation

2.1 Covariant Form of the Dirac Equation

It is necessary that the Dirac equation and the continuity equation upon which its physical interpretation rests be covariant under Lorentz transformations. Let us first review what is meant by a Lorentz transformation.[1] Two observers O and O' who are in different inertial reference frames will describe the same physical event with the different space-time coordinates. The rule which relates the coordinates x^μ with which observer O describes the event to the coordinates $(x^\mu)'$ used by observer O' to describe the same event is given by the Lorentz transformation between the two sets of coordinates:

$$(x^\nu)' = \sum_{\mu=0}^{3} a^\nu{}_\mu x^\mu \equiv a^\nu{}_\mu x^\mu \tag{2.1}$$

It is a linear homogeneous transformation, and the coefficients $a^\nu{}_\mu$ depend only upon the relative velocities and spatial orientations of the two reference frames of O and O'. The basic invariant of the Lorentz transformation is the proper time interval

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = dx^\mu\, dx_\mu \tag{2.2}$$

This is derived from the physical observation that the velocity of light in vacuo is the same in all Lorentz frames. Equations (2.1) and (2.2) lead to the relation on the transformation coefficients

$$a_\mu{}^\nu a^\mu{}_\sigma = \delta^\nu{}_\sigma \tag{2.3}$$

Equations (2.1) and (2.3) serve as defining relations for both proper and improper Lorentz transformations. In the former case the determinant of the transformation coefficients satisfies the relation

$$\det |a| = +1$$

Proper Lorentz transformations can be built up by an infinite succession of infinitesimal transformations. They include transformations to coordinates in relative motion along any spatial direction as well as ordinary three-dimensional rotations. The improper Lorentz transformations are the discrete transformations of space inversion and of time inversion. They cannot be built up from a succession of infinitesimal ones. Their transformation coefficients satisfy the

[1] W. Pauli, "Theory of Relativity," Pergamon Press, New York, 1958. "The Principle of Relativity," collected papers of H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, Dover Publications, Inc., New York, 1923 reissue.

relation

$$\det|a| = -1$$

in both cases.

Our task is to construct a correspondence relating a given set of observations of a Dirac particle made by observers O and O' in their respective reference frames. In other words, we seek a transformation law relating the wave functions $\psi(x)$ and $\psi'(x')$ used by observers O and O', respectively. This transformation law is a rule which allows O' to compute $\psi'(x')$ if given $\psi(x)$. According to the requirement of Lorentz covariance, this transformation law must lead to wave functions which are solutions of Dirac equations of the same form in the primed as well as unprimed reference frame. This form invariance of the Dirac equation expresses the Lorentz invariance of the underlying energy-momentum connection

$$p_\mu p^\mu = m^2c^2$$

upon which the considerations of Chap. 1 were based.

In discussing covariance it is desirable to express the Dirac equation in a four-dimensional notation which preserves the symmetry between ct and x^i. To this end we multiply (1.13) by β/c and introduce the notation

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha_i \qquad i = 1, 2, 3$$

This gives

$$i\hbar\left(\gamma^0 \frac{\partial}{\partial x^0} + \gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3}\right)\psi - mc\psi = 0 \qquad (2.4)$$

The new matrices γ^μ provide an elegant restatement of the commutation relations (1.16)

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}1 \qquad (2.5)$$

where 1 is the 4×4 unit matrix and hereafter will not be explicitly indicated. It is clear from their definition that the γ^i are antihermitian, with $(\gamma^i)^2 = -1$, and that γ^0 is hermitian. In the representation (1.17) they have the form

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \qquad \gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (2.6)$$

It is convenient to introduce the Feynman dagger, or slash, notation:

$$A\!\!\!/ = \gamma^\mu A_\mu = g_{\mu\nu}\gamma^\mu A^\nu = \gamma^0 A^0 - \boldsymbol{\gamma} \cdot \mathbf{A}$$

and in particular

$$\not\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} = \frac{\gamma^0}{c}\frac{\partial}{\partial t} + \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}$$

Equation (2.4) then abbreviates to

$$(i\hbar\not\nabla - mc)\psi = 0 \tag{2.7}$$

or, with $p^\mu = i\hbar \dfrac{\partial}{\partial x_\mu}$,

$$(\not p - mc)\psi = 0 \tag{2.8}$$

Addition of the electromagnetic interaction according to the "minimal" substitution (1.25) gives

$$\left(\not p - \frac{e\not A}{c} - mc\right)\psi = 0$$

This in no way influences considerations of covariance, because both p^μ and A^μ, and hence their difference, are four-vectors.

2.2 Proof of Covariance

In order to establish Lorentz covariance of the Dirac equation, we must satisfy two requirements. The first is that there must be an explicit prescription which allows observer O', given the $\psi(x)$ of observer O, to compute the $\psi'(x')$ which describes to O' the same physical state. Second, according to the relativity principle, $\psi'(x')$ will be a solution of an equation which takes the form of (2.7) in the primed system

$$\left(i\hbar\tilde{\gamma}^\mu \frac{\partial}{\partial x^{\mu\prime}} - mc\right)\psi'(x') = 0$$

The $\tilde{\gamma}^\mu$ satisfy the anticommutation relations (2.5); therefore $\tilde{\gamma}^{0\dagger} = \tilde{\gamma}^0$ and $\tilde{\gamma}^{i\dagger} = -\tilde{\gamma}^i$ as required for a hermitian hamiltonian. As may be shown by a lengthy algebraic proof,[1] all such 4×4 matrices $\tilde{\gamma}^\mu$ are equivalent up to a unitary transformation U:

$$\tilde{\gamma}_\mu = U^\dagger \gamma_\mu U \qquad U^\dagger = U^{-1}$$

[1] See R. H. Good, Jr., *Rev. Mod. Phys.*, **27**, 187 (1955), especially Sec. III, p. 190.

and so we drop the distinction between $\tilde{\gamma}^\mu$ and γ^μ and write

$$(\not{p}' - mc)\psi'(x') = 0 \tag{2.9}$$

with
$$\not{p}' = i\hbar\gamma^\mu \frac{\partial}{\partial x^{\mu\prime}}$$

We ask that the transformation between ψ and ψ' be linear, since both the Dirac equation and the Lorentz transformation (2.1) of the coordinates are themselves linear. We introduce it in the form

$$\psi'(x') = \psi'(ax) = S(a)\psi(x) = S(a)\psi(a^{-1}x') \tag{2.10}$$

where $S(a)$ is a 4×4 matrix which operates upon the four-component column vector $\psi(x)$. It depends upon the relative velocities and spatial orientations of O and O'. S must have an inverse, so that if O knows $\psi'(x')$ which O' uses to describe his observations of a given physical state, he may construct his own wave function $\psi(x)$

$$\psi(x) = S^{-1}(a)\psi'(x') = S^{-1}(a)\psi'(ax) \tag{2.11}$$

We could equally well write, using (2.10),

$$\psi(x) = S(a^{-1})\psi'(ax)$$

which provides the identification

$$S(a^{-1}) = S^{-1}(a)$$

The main problem is to find S. It must satisfy (2.10) and (2.11). If S exists, observer O', given $\psi(x)$ by O, may construct $\psi'(x')$ using (2.10).

By reexpressing the Dirac equation (2.7) of O in terms of $\psi'(x')$ with the aid of (2.11), O' could then check whether $\psi'(x')$ satisfies his own equation (2.9). He would find after left-multiplication by $S(a)$

$$\left[i\hbar S(a)\gamma^\mu S^{-1}(a)\frac{\partial}{\partial x^\mu} - mc\right]\psi'(x') = 0$$

Using (2.1) to write

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = a^\nu{}_\mu \frac{\partial}{\partial x'^\nu}$$

the primed equation is found to be

$$\left[i\hbar S(a)\gamma^\mu S^{-1}(a)a^\nu{}_\mu \frac{\partial}{\partial x^{\nu\prime}} - mc\right]\psi'(x') = 0$$

This is form-invariant, that is, identical with (2.9), provided an S can be found which has the property

$$S(a)\gamma^\mu S^{-1}(a)a^\nu{}_\mu = \gamma^\nu$$

or equivalently

$$a^\nu{}_\mu \gamma^\mu = S^{-1}(a)\gamma^\nu S(a) \tag{2.12}$$

Equation (2.12) is the fundamental relation determining S. In seeking S we are seeking a solution to (2.12). Once we show that (2.12) has a solution and find it, the covariance of the Dirac equation is established. By way of terminology, a wave function transforming according to (2.10) and (2.12) is a four-component Lorentz spinor. We anticipate that S will present novel features not found in tensor calculus, since bilinear forms in ψ such as the probability current (1.20) are expected to form four-vectors.

We first construct S for an infinitesimal proper Lorentz transformation

$$a^\nu{}_\mu = g^\nu{}_\mu + \Delta\omega^\nu{}_\mu \tag{2.13a}$$

with

$$\Delta\omega^{\nu\mu} = -\Delta\omega^{\mu\nu} \tag{2.13b}$$

according to Eq. (2.3) for an invariant proper time interval. Each of the six independent nonvanishing $\Delta\omega^{\mu\nu}$ generates an infinitesimal Lorentz transformation,

$$\Delta\omega^{01} = \Delta\beta$$

for a transformation to a coordinate system moving with a velocity $c\,\Delta\beta$ along the x direction,

$$\Delta\omega^1{}_2 = -\Delta\omega^{12} = \Delta\varphi$$

for a rotation through an angle $\Delta\varphi$ about the z axis, and so forth.

Expanding S in powers of $\Delta\omega^{\nu\mu}$ and keeping only the linear term in the infinitesimal generators, we write

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu} \qquad \text{and} \qquad S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu} \tag{2.14}$$

with

$$\sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

by (2.13b). Each of the six coefficients $\sigma_{\mu\nu}$ is a 4×4 matrix, as are the transformation S and the unit matrix 1. Inserting (2.13) and (2.14) into (2.12) and keeping first-order terms in $\Delta\omega^{\mu\nu}$, we find

$$\Delta\omega^\nu{}_\mu\gamma^\mu = -\frac{i}{4}(\Delta\omega)^{\alpha\beta}(\gamma^\nu\sigma_{\alpha\beta} - \sigma_{\alpha\beta}\gamma^\nu)$$

04/13/2006 07:36 FAX 1 510 643 8497 Physics Dept UC Berkeley 007/012

From the antisymmetry of the generators $\Delta\omega^{\mu\nu}$ there follows

$$2i[g^{\nu}{}_{\alpha}\gamma_{\beta} - g^{\nu}{}_{\beta}\gamma_{\alpha}] = [\gamma^{\nu},\sigma_{\alpha\beta}] \tag{2.15}$$

The problem of establishing proper Lorentz covariance of the Dirac equation is now reduced to that of finding six matrices $\sigma_{\alpha\beta}$ which satisfy (2.15). The simplest guess to make is an antisymmetric product of two matrices, and directly we find, using (2.5), that

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu},\gamma_{\nu}] \tag{2.16}$$

is the desired matrix. According to (2.14), S for an infinitesimal Lorentz transformation is given by

$$S = 1 + \frac{1}{8}[\gamma_{\mu},\gamma_{\nu}]\,\Delta\omega^{\mu\nu} = 1 - \frac{i}{4}\sigma_{\mu\nu}\,\Delta\omega^{\mu\nu} \tag{2.17}$$

We now complete our task by constructing the finite proper transformations by a succession of infinitesimal ones. First, to build up (2.1) from (2.13), we write

$$\Delta\omega^{\nu}{}_{\mu} = \Delta\omega(I_n)^{\nu}{}_{\mu} \tag{2.18}$$

where $\Delta\omega$ is the infinitesimal parameter, or "angle of rotation" about an axis in the direction labeled n, and I_n is the 4×4 (in space-time) matrix of coefficients for a unit Lorentz rotation about this axis. ν and μ label row and column respectively. Thus for a transformation to a primed system in motion along the x axis with an infinitesimal velocity $c\,\Delta\omega = c\,\Delta\beta$

$$I^{\nu}{}_{\mu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.19}$$

so that

$$I^{0}{}_{1} = I^{1}{}_{0} = -I^{01} = +I^{10} = -1$$

Using the algebraic property of $I^{\nu}{}_{\mu}$, that

$$I^{2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad I^{3} = +I$$

we can write the finite transformation for uniform relative x-axis motion as

$$\begin{aligned} x^{\nu\prime} &= \lim_{N\to\infty} \left(g + \frac{\omega}{N} I\right)^{\nu}_{\ \alpha_1} \left(g + \frac{\omega}{N} I\right)^{\alpha_1}_{\ \alpha_2} \cdots x^{\alpha_N} \\ &= (e^{\omega I})^{\nu}{}_{\mu} x^{\mu} \\ &= (\cosh \omega I + \sinh \omega I)^{\nu}{}_{\mu} x^{\mu} \\ &= (1 - I^2 + I^2 \cosh \omega + I \sinh \omega)^{\nu}{}_{\mu} x^{\mu} \end{aligned}$$

For the individual components this gives

$$\begin{bmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{bmatrix} = \begin{bmatrix} \cosh \omega & -\sinh \omega & 0 & 0 \\ -\sinh \omega & \cosh \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{2.20}$$

or

$$\begin{aligned} x^{0\prime} &= (\cosh \omega)(x^0 - \tanh \omega\, x^1) \\ x^{1\prime} &= (\cosh \omega)(x^1 - \tanh \omega\, x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned} \tag{2.21}$$

where $\tanh \omega = \beta$ and $\cosh \omega = \dfrac{1}{\sqrt{1 - \beta^2}}$

relate the Lorentz rotation angle ω with the relative velocity $c\beta$.

This result can be generalized to include motion along any direction or spatial rotation about any axis. The six matrices $I^{\nu}{}_{\mu}$ generating the six independent Lorentz rotations are the four-dimensional generalizations of the three-dimensional space rotations familiar in the nonrelativistic theory.

Turning now to the construction of a finite spinor transformation S, we have from (2.14) and (2.18)

$$\begin{aligned} \psi'(x') = S\psi(x) &= \lim_{N\to\infty} \left(1 - \frac{i}{4}\frac{\omega}{N} \sigma_{\mu\nu} I_n^{\mu\nu}\right)^N \psi(x) \\ &= \exp\left(-\frac{i}{4}\omega\sigma_{\mu\nu} I_n^{\mu\nu}\right)\psi(x) \end{aligned} \tag{2.22}$$

Specializing again to the transformation (2.19) we have

$$\psi'(x') = e^{-(i/2)\omega\sigma_{01}}\psi(x) \tag{2.23}$$

where x' and x are related by (2.21).

Similarly, for a rotation through an angle φ about the z axis, $I^{12} = -I^{21} = -1$ and

$$\psi'(x') = e^{(i/2)\varphi\sigma^{12}}\psi(x) \tag{2.24}$$

where

$$\sigma^{12} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}$$

in the representation (1.17), with

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

the Pauli 2×2 matrix. We recognize the similarity of (2.24) with the form of rotation of a two-component Pauli spinor

$$\varphi'(x') = e^{(i/2)\boldsymbol{\omega}\cdot\boldsymbol{\sigma}}\varphi(x) \tag{2.25}$$

The covariant "angle" variables $\omega^{\mu\nu}$ in (2.18) are associated with the Lorentz transformation in the same sense that the rotation angle and direction in $\boldsymbol{\omega}$ are for the three-dimensional rotation. The appearance of half-angles in (2.24), as in (2.25), is an expression of the double-valuedness of the spinor law of rotation; it takes a rotation of 4π radians to return $\psi(x)$ to its original value. Because of this, physical observables in the Dirac theory must be bilinear, or an even power in $\psi(x)$.

For spatial rotations, $S = S_R$ is unitary, since the σ_{ij} are hermitian, and

$$S_R^\dagger = e^{-(i/4)\sigma^{\dagger ij}\omega_{ij}} = e^{-(i/4)\sigma^{ij}\omega_{ij}} = S_R^{-1}$$

This is not true for transformations to a moving coordinate system $S = S_L$. For instance, for the transformation (2.23)

$$S_L = e^{-(i/2)\omega\sigma_{01}} = e^{-(\omega/2)\alpha_1} = S_L^\dagger \neq S_L^{-1}$$

However, S_L does have the property

$$S_L^{-1} = \gamma_0 S_L^\dagger \gamma_0$$

found by expanding S_L in a power series. Since $[\gamma_0,\sigma^{ij}] = 0$, this can be generalized to include rotations

$$S^{-1} = \gamma_0 S^\dagger \gamma_0 \tag{2.26}$$

The continuity equation is also covariant. The probability current (1.21) and (1.22), in the notation of (2.4), is

$$j^\mu(x) = c\psi^\dagger(x)\gamma^0\gamma^\mu\psi(x)$$

and under (2.1) transforms to

$$\begin{aligned} j^{\mu\prime}(x') &= c\psi'^{\dagger}(x')\gamma^0\gamma^\mu\psi'(x') \\ &= c\psi^\dagger(x)S^\dagger\gamma_0\gamma^\mu S\psi(x) \\ &= c\psi^\dagger(x)\gamma_0 S^{-1}\gamma^\mu S\psi(x) \\ &= ca^\mu{}_\nu\psi^\dagger(x)\gamma_0\gamma^\nu\psi(x) \\ &= a^\mu{}_\nu j^\nu(x) \end{aligned} \tag{2.27}$$

Evidently $j^\mu(x)$ is a Lorentz four-vector and the continuity equation

$$\frac{\partial j^\mu(x)}{\partial x^\mu} = 0$$

is invariant. Also, the probability density $j^0(x) = c\rho(x)$ transforms as the time component of a conserved four-vector. This is the desired result noted in Sec. 1.3 for an invariant probability.

Because the combination $\psi^\dagger\gamma_0$ in (2.27) occurs so often, it is dignified by a new notation

$$\bar{\psi}(x) = \psi^\dagger\gamma_0 \tag{2.28}$$

where $\bar{\psi}(x)$ is known as the adjoint spinor. Its Lorentz transformation property is given by

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1} \tag{2.29}$$

2.3 Space Reflection

We now expand our outlook to take into account the existence of the improper Lorentz transformation of space reflection

$$\mathbf{x}' = -\mathbf{x} \qquad t' = t$$

Again covariance requires a solution of (2.12), but in this case we cannot build it up from the infinitesimal transformations. However, it is easy enough to solve (2.12) directly. The transformation matrix is

$$a^\nu{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = g^{\nu\mu} \tag{2.30}$$

Denoting $S = P$ for the coordinate reflection, (2.12) becomes

$$P^{-1}\gamma^\nu P = g^{\nu\nu}\gamma^\nu \tag{2.31}$$

which is satisfied by

$$P = e^{i\varphi}\gamma_0 \tag{2.32}$$

The phase factor is of no physical interest here and may be narrowed down to the four choices ± 1, $\pm i$ if we require that four reflections return the spinor to itself in analogy with a rotation through 4π radians. P in (2.32) evidently is unitary, $P^{-1} = P^{\dagger}$, and satisfies (2.26) as well. Equation (2.32) tells us that

$$\psi'(x') = \psi'(-\mathbf{x},t) = e^{i\varphi}\gamma_0\psi(\mathbf{x},t) \tag{2.33}$$

In the nonrelativistic limit ψ approaches an eigenstate of P, and by (1.24) and (2.6) the positive- and negative-energy states at rest have opposite eigenvalues, or *intrinsic parities.*

The discussion of the other improper transformations, such as time reversal, is more involved; it is given in Chap. 5.

2.4 Bilinear Covariants

By forming products of the γ matrices it is possible to construct 16 linearly independent 4×4 matrices $\Gamma^n_{\alpha\beta}$ which appear often in applications of the Dirac theory. These are

$$\Gamma^S = 1 \qquad \Gamma^V_\mu = \gamma_\mu \qquad \Gamma^T_{\mu\nu} = \sigma_{\mu\nu}$$
$$\Gamma^P = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5 \equiv \gamma^5 \qquad \Gamma^A_\mu = \gamma_5\gamma_\mu \tag{2.34}$$

By using the anticommutation relations (2.5) the Γ^n are readily established to be linearly independent by the following argument:

1. For each Γ^n, $(\Gamma^n)^2 = \pm 1$.
2. For each Γ^n except Γ^S, there exists a Γ^m such that

$$\Gamma^n\Gamma^m = -\Gamma^m\Gamma^n$$

From this it follows that the trace of Γ^n vanishes:

$$\pm \operatorname{Tr} \Gamma^n = \operatorname{Tr} \Gamma^n(\Gamma^m)^2 = -\operatorname{Tr} \Gamma^m\Gamma^n\Gamma^m = -\operatorname{Tr} \Gamma^n(\Gamma^m)^2 = 0$$

3. Given Γ^a and Γ^b, $a \neq b$, there exists a $\Gamma^n \neq \Gamma^S$ such that

$$\Gamma^a\Gamma^b = \Gamma^n$$

This follows by direct inspection of the Γ's.

4. Suppose there exist numbers a_n such that

$$\sum_n a_n\Gamma^n = 0$$

Then multiply by $\Gamma^m \neq \Gamma^S$ and take the trace; using (3), we find $a_m = 0$. If $\Gamma^m = \Gamma^S$, we find $a_s = 0$, and all coefficients vanish.

This establishes the linear independence of the Γ^n. It follows that any 4×4 matrix can be written in terms of the Γ^n.

We may now write down the Lorentz transformation properties of the bilinear forms $\bar{\psi}(x)\Gamma^n\psi(x)$ constructed from the 16 Γ^n. We need only the observation that

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0 \tag{2.35}$$

and therefore

$$[\gamma_5,\sigma_{\mu\nu}] = 0$$

or

$$[S,\gamma_5] = 0 \tag{2.36}$$

for all proper Lorentz transformations. As a special case of (2.35)

$$P\gamma_5 = -\gamma_5 P \tag{2.37}$$

Carrying out calculations similar to (2.27) we find:

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x)$$
a scalar

$$\bar{\psi}'(x')\gamma_5\psi'(x') = \bar{\psi}(x)S^{-1}\gamma_5 S\psi(x) = \det|a|\bar{\psi}(x)\gamma_5\psi(x)$$
a pseudoscalar

$$\bar{\psi}'(x')\gamma^\nu\psi'(x') = a^\nu{}_\mu\bar{\psi}(x)\gamma^\mu\psi(x)$$
a vector

$$\bar{\psi}'(x')\gamma_5\gamma^\nu\psi'(x') = \det|a|a^\nu{}_\mu\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$$
a pseudovector

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = a^\mu{}_\alpha a^\nu{}_\beta\bar{\psi}(x)\sigma^{\alpha\beta}\psi(x)$$
a second-rank tensor (2.38)

Problems

1. Verify (2.26).

2. Verify the transformation laws given in (2.38).

3. Given a free-particle spinor $u(p)$, construct $u(p+q)$ for $q_\mu \to 0$, with $p\cdot q \to 0$, in terms of $u(p)$ by making a Lorentz transformation.

4. Show that there exist four 4×4 matrices Γ^μ such that

$$\mathrm{Re}\,\Gamma^\mu_{\alpha\beta} = 0$$

$$\{\Gamma_\mu,\Gamma_\nu\} = 2g_{\mu\nu}$$

$$\left[i\Gamma_\mu \frac{\partial}{\partial x_\mu} - m\right]\psi(x) = 0$$

that is, the Dirac equation is real.