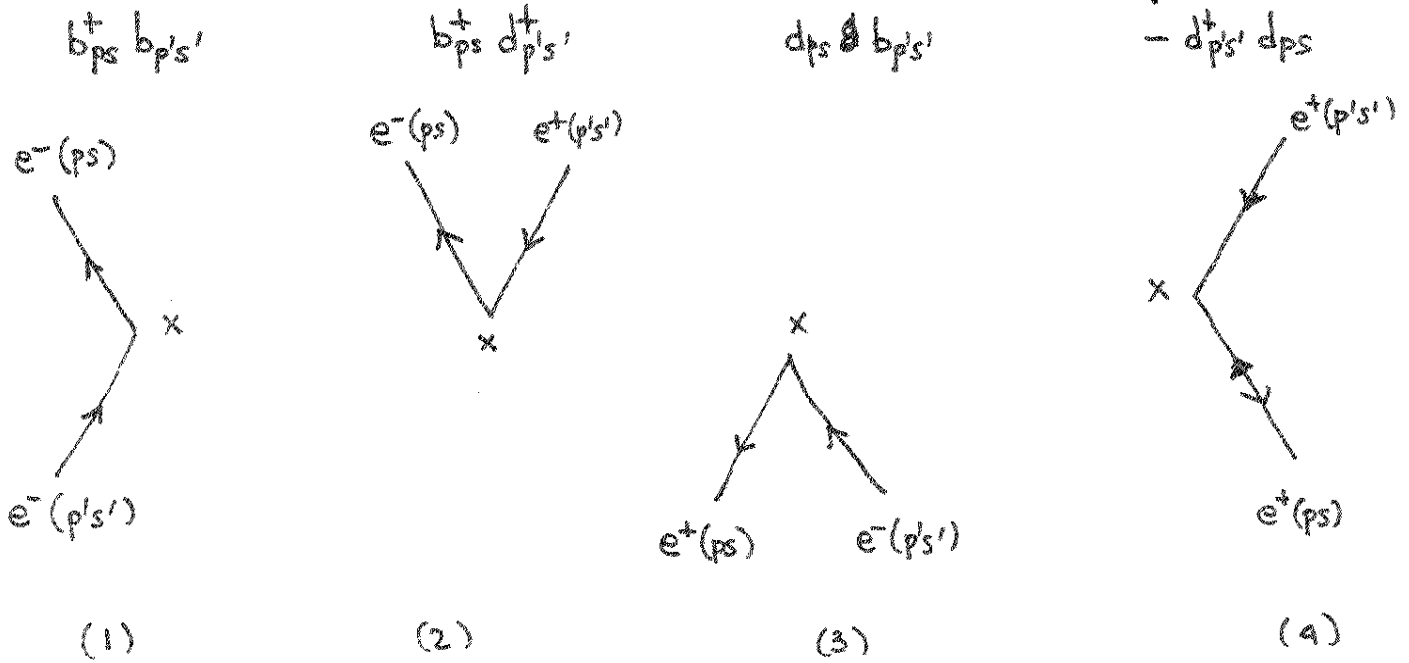


notice the sign change from
normal ordering

①



Case (1) is scattering of an electron by the external field A_μ , indicated by the X . Case (2) is creation of an e^+e^- pair in the field. Case (3) is the annihilation of an e^+e^- pair. Case (4) is the scattering of a positron. Arrows go in the direction of time for an electron, and backwards for a positron, following the ideas of Stückelberg and Feynman.

Now let's work through an example, the scattering of an electron in an electrostatic potential $\Phi(\vec{x})$. Thus,

~~$$A^\mu = (\Phi, \vec{0})$$~~

$$A^\mu = (\Phi, \vec{0}),$$

$$\gamma^\mu A_\mu = \not{A} = \gamma^0 \Phi(\vec{x}).$$

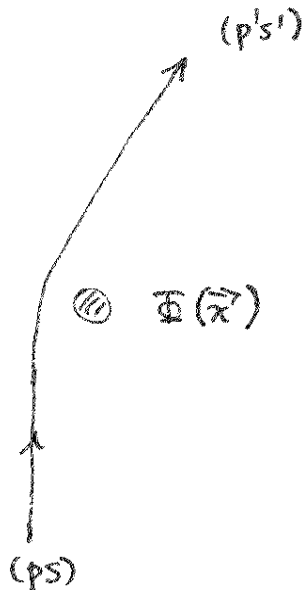
The initial and final states are

②

$$|i\rangle = b_{ps}^+ |0\rangle$$

$$|n\rangle = b_{p's'}^+ |0\rangle$$

an electron in state (ps) is scattered into state $(p's')$. The picture in 3D space is



It could be scattering by a nucleus (relativistic Rutherford scattering, also called Mott scattering.) The matrix element is (with $q = -e$ for electrons)

$$M = \langle n | H_1 | i \rangle = \langle 0 | b_{p's'} H_1 b_{ps}^+ | 0 \rangle$$

$$= \frac{-e}{V} \int d^3\vec{x} \langle 0 | b_{p's'} \cdot \sum_{p_1 s_1} \sqrt{\frac{m}{E_1}} \left(b_{p_1 s_1}^+ \bar{u}_{p_1 s_1} e^{-i\vec{p}_1 \cdot \vec{x}} + d_{p_1 s_1} \bar{v}_{p_1 s_1} e^{i\vec{p}_1 \cdot \vec{x}} \right)$$

$$\times \gamma^0 \Phi(\vec{x}) \sum_{p_2 s_2} \sqrt{\frac{m}{E_2}} \left(b_{p_2 s_2} u_{p_2 s_2} e^{i\vec{p}_2 \cdot \vec{x}} + d_{p_2 s_2}^+ v_{p_2 s_2} e^{-i\vec{p}_2 \cdot \vec{x}} \right) :$$

$$\times b_{ps}^+ | 0 \rangle$$

Here we use the Fourier series for $\psi, \bar{\psi}$:

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left(b_{ps} u_{ps} e^{i\vec{p}\cdot\vec{x}} + d_{ps}^\dagger v_{ps} e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\bar{\psi}(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left(b_{ps}^\dagger \bar{u}_{ps} e^{-i\vec{p}\cdot\vec{x}} + d_{ps} \bar{v}_{ps} e^{+i\vec{p}\cdot\vec{x}} \right)$$

Because of the anticommutation relations among the b's and d's, the only terms that survive in the two Fourier series in the matrix element M is the $b^\dagger b$ term with $(p_1, s_1) = (p', s')$ and $(p_2, s_2) = (p, s)$. That is,

$$\langle 0 | b_{p's'} b_{p_2 s_2}^\dagger b_{p_1 s_1}^\dagger b_{ps} | 0 \rangle$$

anticommute

$$\begin{aligned} & \rightarrow = - \langle 0 | b_{p's'} b_{p_2 s_2}^\dagger b_{ps}^\dagger b_{p_1 s_1} | 0 \rangle + \delta_{p_2 p} \delta_{s_2 s} \langle 0 | b_{p's'} b_{p_1 s_1}^\dagger | 0 \rangle \\ & \rightarrow = - \langle 0 | b_{p_2 s_2}^\dagger b_{p's'} | 0 \rangle + \delta_{p' p_1} \delta_{s' s_1} \end{aligned}$$

kills

anticommute.

All other terms $b^\dagger d^\dagger$ etc. give 0, i.e., only the electron scattering diagram (1) above contributes. Thus,

$$M = \frac{-e}{V} \int d^3\vec{x} \sqrt{\frac{m^2}{EE'}} (\bar{u}_{p's'} \gamma^0 u_{ps}) e^{-i(\vec{p}' - \vec{p})\cdot\vec{x}} \Phi(\vec{x})$$

(4)

$$= \frac{-e}{V} (2\pi)^{3/2} \sqrt{\frac{m^2}{EE'}} \tilde{\Phi}(\vec{q}) (\bar{u}_{p's'} \gamma^0 u_{ps}),$$

where

$$\tilde{\Phi}(\vec{q}) = \int \frac{d^3\vec{x}}{(2\pi)^{3/2}} e^{-i\vec{q}\cdot\vec{x}} \Phi(\vec{x}), \quad \text{F.T. of potential } \Phi(\vec{x}),$$

and where

$$\vec{q} = \vec{p}' - \vec{p} \quad (\text{3-momentum transfer in scattering}).$$

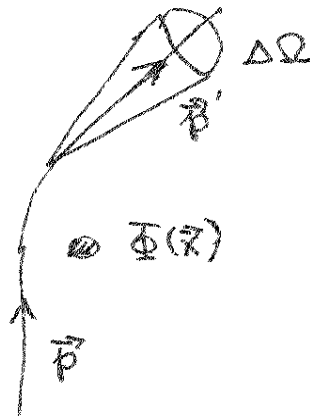
To get the diff. cross section, first compute the incident flux of electrons,

$$J_{inc} = \frac{1}{V} \frac{p}{E},$$

where $1/V = \# \text{ electrons / vol}$, velocity = p/E (relativistic expression).

Then

$$\frac{d\sigma}{d\Omega} = \frac{1}{J_{inc}} \frac{1}{t} \frac{1}{\Delta\Omega} \sum_{\vec{p}' \in \text{cone}} |C_n(t)|^2$$



where

$$|C_n(t)|^2 = 2\pi t \Delta_{\perp} (E' - E) |M|^2$$

5

or,

$$\frac{d\sigma}{d\Omega} = V \frac{1}{\frac{E}{p}} \frac{1}{t} \frac{1}{\Delta\Omega} \left(\sum_{\vec{p}' \in \text{cone}} 2\pi t \Delta_t(E'-E) \frac{e^2}{V^2} (2\pi)^3 \left| \tilde{\Phi}(\vec{q}) \right|^2 \left| \bar{u}_{p's'} \gamma^0 u_{ps} \right|^2 \right)$$

$\lim_{V \rightarrow \infty} \frac{V}{(2\pi)^3} \Delta\Omega \int_0^\infty p'^2 dp'$

Also $\lim_{t \rightarrow \infty} \Delta_t(E'-E) = \delta(E'-E) = \frac{\delta(p'-p)}{p/E}$.

↙ magnitudes of \vec{p}', \vec{p}

Clean it up, get

$$\frac{d\sigma}{d\Omega} = 2\pi e^2 m^2 \left| \tilde{\Phi}(\vec{q}) \right|^2 \left| \bar{u}_{p's'} \gamma^0 u_{ps} \right|^2.$$

Note that when $p'=p$ (from δ -fn), we have $E'=E$, and the 4-momenta are

$$\begin{aligned} \text{4-momenta.} & \begin{cases} \vec{p} = (E, \vec{p}) \\ \vec{p}' = (E, \vec{p}') \end{cases} \end{aligned}$$

Now all we need to do is the contraction in Dirac spin space, $\bar{u}_{p's'} \gamma^0 u_{ps}$. The answer depends on the initial and final spin states s, s' . In the NR limit electron scattering by an electrostatic potential is independent of the spin, but at higher velocities the spin becomes more important.

For simplicity we will assume that the initial beam is

⑥

unpolarized and we don't care about the spin of the final electron. Then we must average over initial states and sum over final states, i.e.,

$$\begin{aligned}
 & |\bar{u}_{p's'} \gamma^0 u_{ps}|^2 \rightarrow \frac{1}{2} \sum_{ss'} |\bar{u}_{p's'} \gamma^0 u_{ps}|^2 \\
 &= \frac{1}{2} \sum_{ss'} \bar{u}_{p's'} \gamma^0 u_{ps} \bar{u}_{ps} \gamma^0 u_{p's'} \\
 &= \frac{1}{2} \sum_{ss'} \text{tr} \left[\gamma^0 u_{ps} \bar{u}_{ps} \gamma^0 u_{p's'} \bar{u}_{p's'} \right] \\
 &= \frac{1}{2} \text{tr} \left[\gamma^0 \left(\sum_s u_{ps} \bar{u}_{ps} \right) \gamma^0 \left(\sum_{s'} u_{p's'} \bar{u}_{p's'} \right) \right].
 \end{aligned}$$

The two sums are projectors onto positive energy spinors, i.e.,

$$\Delta_+(p) = \frac{\not{p} + m}{2m} = \sum_s u_{ps} \bar{u}_{ps}$$

$$\Delta_-(p) = -\frac{\not{p} - m}{2m} = -\sum_s v_{ps} \bar{v}_{ps}$$

see p. 33 of Bjorken + Drell. So we must compute

$$\frac{1}{2} \text{tr} \left[\gamma^0 \left(\frac{\not{p} + m}{2m} \right) \gamma^0 \left(\frac{\not{p}' + m}{2m} \right) \right].$$

⑦

Now we need some rules for taking the traces of products of γ matrices. First we present the rules, then we prove them.

Rules:

1. $\text{tr } 1 = 4$
2. $\text{tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$
3. $\text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4 (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha})$
4. $\text{tr}(\text{any odd \# } \gamma\text{'s}) = 0.$

⊗ Alternative versions of 2, 3:

$$2'. \text{Tr}(a b) = 4 a \cdot b$$

$$3'. \text{Tr}(a b c d) = 4 [(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$$

where a, b, c, d are 4 vectors and $a \cdot b = a^\mu b_\mu$ etc.

Proof of 1 is trivial.

Proof of 2:

$$\begin{aligned} \text{tr}(\gamma^\mu \gamma^\nu) &= 2 g^{\mu\nu} \text{tr}(1) - \underbrace{\text{tr}(\gamma^\nu \gamma^\mu)} \\ &\quad \downarrow \\ &\quad \text{anticomm.} \end{aligned}$$

↖ = $\text{tr}(\gamma^\mu \gamma^\nu) = \text{LHS.}$
cyclic perm.

So,

$$2 \text{tr}(\gamma^\mu \gamma^\nu) = 2 g^{\mu\nu} \times 4, \text{ QED.}$$

Proof of 3:

$$\text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 2g^{\mu\nu} \text{tr}(\gamma^\alpha \gamma^\beta) - \text{tr}(\gamma^\nu \gamma^\mu \gamma^\alpha \gamma^\beta)$$

$$\rightarrow = -2g^{\mu\alpha} \text{tr}(\gamma^\nu \gamma^\beta) + \text{tr}(\gamma^\nu \gamma^\alpha \gamma^\mu \gamma^\beta)$$

$$\rightarrow = +2g^{\mu\beta} \text{tr}(\gamma^\nu \gamma^\alpha) - \text{tr}(\gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\mu)$$

$$\rightarrow = -\text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = -\text{LHS.}$$

from which (3) follows. Obviously any trace of $2n$ γ 's can be reduced to traces of $2n-2$ γ 's.

Proof of 4 uses properties of $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, namely,

$$\gamma_5^2 = 1$$

$$\{\gamma_5, \gamma^\mu\} = 0.$$

Example of 3 γ 's:

3 anticommutes.

$$\text{tr}(\gamma^\mu \gamma^\alpha \gamma^\beta) = \text{tr}(\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_5^2) = \text{tr}(\gamma_5 \gamma^\mu \gamma^\alpha \gamma^\beta \gamma_5)$$

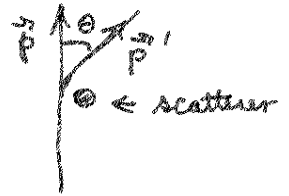
$$= -\text{tr}(\gamma^\mu \gamma^\beta \gamma^\alpha \gamma_5^2) = -\text{tr}(\gamma^\mu \gamma^\beta \gamma^\alpha) = 0$$

(9)

Now we can do traces.

$$\frac{1}{2} \text{tr} \left[\gamma^0 \left(\frac{\not{p} + m}{2m} \right) \gamma^0 \left(\frac{\not{p}' + m}{2m} \right) \right] = \frac{1}{8m^2} \text{tr} \left[\gamma^0 \not{p} \gamma^0 \not{p}' + m^2 \gamma^0 \gamma^0 \right]$$

$$= \frac{4}{8m^2} \left(\cancel{E E'} - \cancel{(\not{p} \cdot \not{p}') + E' E} + m^2 g^{00} \right)$$



Use

$$g^{00} = 1$$

$$E = E' \quad (\text{conservation of energy})$$

$$p \cdot p' = E^2 - \vec{p} \cdot \vec{p}'$$

$$m^2 = E^2 - |\vec{p}|^2$$

$$\text{and note } |\vec{p}| = |\vec{p}'|,$$

$$\rightarrow = \frac{1}{2m^2} \left(2E^2 - |\vec{p}|^2 + \vec{p} \cdot \vec{p}' \right) = \frac{1}{2m^2} \left(2E^2 - |\vec{p}|^2 (1 - \cos\theta) \right)$$

$$= \frac{1}{m^2} \left(E^2 - |\vec{p}|^2 \sin^2\theta/2 \right) = \frac{E^2}{m^2} \left(1 - \beta^2 \sin^2\theta/2 \right), \quad \beta = \frac{v}{c} = \frac{|\vec{p}|}{E}$$

So,

$$\frac{d\sigma}{d\Omega} = 2\pi e^2 E^2 |\tilde{\Phi}(\vec{q})|^2 \left(1 - \beta^2 \sin^2\theta/2 \right)$$

For Coulomb case, $\Phi(\vec{x}) = \frac{Ze}{|\vec{x}|}$, $\tilde{\Phi}(\vec{q}) = \frac{4\pi Ze}{(2\pi)^{3/2} q^2}$,

get

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 e^4 E^2}{|\vec{p}|^4 \sin^4\theta/2} \left(1 - \beta^2 \sin^2\theta/2 \right)$$

(Mott cross section).
(relativistic generalization
of Rutherford)

The term $\beta^2 \sin^2\theta/2$ is of order $(v/c)^2$, obviously a relativistic correction. It is due to the magnetic interactions of the electron

(through its spin) with the potential $\Phi(\vec{x})$. This term is not present in the $d\sigma/d\Omega$ for scattering of spinless bosons.

Now we consider a more sophisticated problem, namely, e^+e^- annihilation. The reaction is

$$e^- + e^+ \rightarrow \gamma\gamma.$$

Although the Feynman diagram for single photon annihilation exists

$$(e^- + e^+ \rightarrow \gamma)$$



this process cannot occur in free space because you cannot conserve both energy and momentum in $e^+ + e^- \rightarrow \gamma$.

So 2-photon annihilation is the simplest that can occur in free space.

We must now quantize the EM field as well as the e^+e^- field.

To do this we return to the ^{level of the} classical Lagrangian, and write

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_{em} + \mathcal{L}_{int},$$

where

$$\mathcal{L}_D = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi \quad (\text{free Dirac particle})$$

$$\mathcal{L}_{em} = \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi} = \frac{E^2 - B^2}{8\pi} \quad (\text{EM field})$$

$$\mathcal{L}_{int} = e \bar{\Psi} \gamma^\mu A_\mu \Psi = -\mathbf{J}_\mu A^\mu \quad (\text{interaction})$$

$$= e \Psi^\dagger \Phi \Psi - e \Psi^\dagger \vec{\alpha} \cdot \vec{A} \Psi$$

where we set $q = -e$. The 3 terms in \mathcal{L} are manifestly Lorentz

invariant, as required by an acceptable relativistic theory.

Here $A^\mu = (\Phi, \vec{A})$. In Coulomb gauge, Φ is not an ^{independent} dyn. variable, (11)
but rather is a fn. of the matter variables, here ψ and $\bar{\psi}$. That is,

$$\Phi(\vec{x}) = \int d^3\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

where $\rho(\vec{x}) = \sum_{\alpha} \psi^\dagger(\vec{x}) \psi(\vec{x})$.

This Lagrangian is interpreted here at the "classical" level, which means 1st quantized Dirac field plus unquantized EM field. The eqns of motion are the Dirac eqn + Maxwell's eqns (coupled), i.e.

$$(i\gamma^\mu \partial_\mu - m)\psi = -e A \psi$$

$$F^{\mu\nu}_{,;\nu} = 4\pi J^\mu = \cancel{4\pi e \bar{\psi} \gamma^\mu \psi} - 4\pi e \bar{\psi} \gamma^\mu \psi.$$

We convert \mathcal{L} to a Hamiltonian. Let π be the field conjugate to ψ and \vec{P} the field conjugate to \vec{A} . Then

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0$$

$$\vec{P} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = \frac{\dot{\vec{A}}}{4\pi}$$

The second eqn. follows from fact that

$$\mathcal{L}_{em} = \frac{E^2 - B^2}{8\pi} = \frac{E_{||}^2}{8\pi} + \frac{E_{\perp}^2 + B^2}{8\pi},$$

where $\vec{E}_{||} = -\nabla\Phi$ and $\vec{E}_{\perp} = -\dot{\vec{A}}$ (in Coulomb gauge). The momenta conjugate to $\bar{\psi}$ and Φ are zero (there is no $\dot{\bar{\psi}}$ or $\dot{\Phi}$ in the

Lagrangian). So for the Hamiltonian we get

(12)

$$\begin{aligned} \mathcal{H} &= \pi \dot{\psi} + \vec{P} \cdot \vec{A} - \mathcal{L} \\ &= \bar{\psi} (i\gamma^0 \partial_0 \psi + \frac{E_{\perp}^2}{4\pi} - \bar{\psi} (i\vec{\alpha} \cdot \nabla - m) \psi \\ &\quad - \frac{E_{\parallel}^2}{8\pi} - \frac{E_{\perp}^2 - B^2}{8\pi} - e \psi^{\dagger} \Phi \psi + e \psi^{\dagger} \vec{\alpha} \cdot \vec{A} \psi \end{aligned}$$

Note,



$$-e \psi^{\dagger} \Phi \psi = \rho(\vec{x}) \Phi(\vec{x}),$$

$$-\frac{E_{\parallel}^2}{8\pi} = -\frac{|\nabla \Phi|^2}{8\pi}.$$

$$e \psi^{\dagger} \vec{\alpha} \cdot \vec{A} \psi = e \bar{\psi} (\vec{\alpha} \cdot \vec{A}) \psi.$$

When we integrate over space we use the fact that

$$\begin{aligned} \int d^3\vec{x} \frac{E_{\parallel}^2}{8\pi} &= \int d^3\vec{x} \frac{|\nabla \Phi|^2}{8\pi} = -\int d^3\vec{x} \Phi \nabla^2 \Phi \\ &= +\frac{1}{2} \int d^3\vec{x} \rho \Phi \end{aligned}$$

~~Thus~~ Thus the Hamiltonian can be written,

$$H = \int d^3\vec{x} \mathcal{H} = \overset{H_0 + H_{em} + H_{int}}{\mathcal{H}_0 + \mathcal{H}_{em} + \mathcal{H}_{int}}$$

where

$$H_0 = \int d^3\vec{x} \psi^{\dagger} (-i\vec{\alpha} \cdot \nabla + m\beta) \psi$$

$$H_{em} = \int d^3\vec{x} \frac{E_{\perp}^2 + B^2}{8\pi}$$

~~What is $\frac{1}{2} \int d^3\vec{x} \frac{E_{\perp}^2 + B^2}{8\pi}$?~~ "H-transverse"

$$H_{int} = H_{coul} + H_T,$$

where

$$\begin{aligned} H_{coul} &= \frac{1}{2} \int d^3\vec{x} \rho \Phi = \frac{1}{2} \int d^3\vec{x} d^3\vec{x}' \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \\ &= \frac{e^2}{2} \int d^3\vec{x} d^3\vec{x}' \frac{\psi^\dagger(\vec{x}) \psi(\vec{x}) \psi^\dagger(\vec{x}') \psi(\vec{x}')}{|\vec{x} - \vec{x}'|} \end{aligned}$$

and

$$H_T = e \int d^3\vec{x} \bar{\psi} (\vec{\gamma} \cdot \vec{A}) \psi.$$

We now quantize this, reinterpreting Φ, ψ, \vec{A} as field operators.

We must normal order so vacuum expectation values will vanish. We expand the free field Hamiltonians in normal modes, but leave the interacting Hamiltonians as spatial integrals. This gives

$$H = H_0 + H_1$$

$$H_0 = H_D + H_{em}$$

$$H_1 = H_{coul} + H_T$$

$$H_D = \int d^3\vec{x} : \psi(\vec{x})^\dagger (-i \vec{\alpha} \cdot \nabla + m\beta) \psi(\vec{x}) : = \sum_{ps} E (b_{ps}^\dagger b_{ps} + d_{ps}^\dagger d_{ps})$$

$$H_{em} = \int d^3\vec{x} : \frac{E_{\perp}^2 + B^2}{8\pi} : = \sum_{\lambda} \omega_{\lambda} a_{\lambda}^\dagger a_{\lambda}$$

$$H_{\text{Coul}} = \frac{e^2}{2} \int d^3\vec{x} d^3\vec{x}' \frac{:\psi^\dagger(\vec{x})\psi(\vec{x})\psi^\dagger(\vec{x}')\psi(\vec{x}') :}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{2} \int d^3\vec{x} d^3\vec{x}' \frac{:\rho(\vec{x})\rho(\vec{x}') :}{|\vec{x}-\vec{x}'|}$$

$$H_T = e \int d^3\vec{x} : \bar{\Psi}(\vec{x}) \vec{\gamma} \cdot \vec{A}(\vec{x}) \Psi(\vec{x}) :$$

It is a guess that this is the correct Hamiltonian describing the electron-position-photon fields in interaction.

Let us now examine the processes engendered by this $H_I = H_{\text{Coul}} + H_T$ in 1st order TDPT. (we start with H_T .) The general structure of a matrix element is

$$\langle f | H_T | i \rangle \sim \int d^3\vec{x} \langle f | : (b^\dagger \dots d) (a \dots a^\dagger) (b \dots d^\dagger) : | i \rangle$$

There are 8 terms, which give the same Feynman diagrams shown on p.3, 5/5/06, except that a photon is attached to the vertex (either created or destroyed). Thus we have



1a



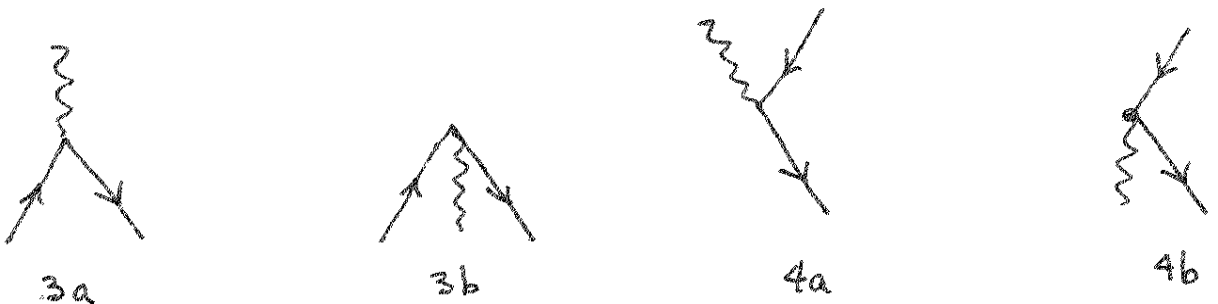
1b



2a



2b



These are the basic Feynman diagrams generated by one application of H_I . There are only 3-point vertices, because H_I is cubic in the field operators ($\bar{\psi} A \psi$). Also, charge is conserved at each vertex.

Return to $e^- e^+$ annihilation. Label initial and final modes, states.

$$\begin{array}{ccc} (ps) & (p's') & \lambda \lambda' \\ e^- + e^+ & \rightarrow & \gamma \gamma \end{array}$$

$$\text{initial state } |i\rangle = b_{ps}^\dagger d_{p's'}^\dagger |0\rangle$$

$$\text{final state } |f\rangle = |n\rangle = a_\lambda^\dagger a_{\lambda'}^\dagger |0\rangle. \quad \text{This process cannot be}$$

accomplished by H_I in 1st order pert. theory, because H_I is capable of creating or destroying one photon, and we must create two photons on going from $|i\rangle$ to $|f\rangle$. But H_I can do it in 2nd order pert. theory. As for H_{coal} , it cannot create any photons at all, since it has no \vec{A} in it. Therefore it does not contribute to pair annihilation at lowest order, and we ignore it henceforth.

Through 2nd order, TDPT gives the transition probability $i \rightarrow n$ as