For the Dirac field, we get

$$H = \int\limits_{box} d^3\vec{x} \ \psi^\dagger \left(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta \right) \psi$$

$$= \int d^3\vec{x} \ \frac{1}{V} \sum_{\substack{PS \\ p's'}} \sqrt{\frac{m^2}{EE'}} \left(b^*_{p's'} u(p's')^\dagger e^{-i\vec{p}'\cdot\vec{x}} + c^*_{p's'} v(p's') e^{+i\vec{p}'\cdot\vec{x}} \right)$$

$$\times \left(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta \right) \left(b_{ps} u(ps) e^{i\vec{p}\cdot\vec{x}} + c_{ps} v(ps) e^{-i\vec{p}\cdot\vec{x}} \right)$$

When the Dirac Hamiltonian $(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)$ acts on free particle solns, it brings out either E or $-E$ (for pos + neg. energy solns). Thus the last 2 factors become

$$E \left(b_{ps} u(ps) e^{i\vec{p}\cdot\vec{x}} - c_{ps} v(ps) e^{-i\vec{p}\cdot\vec{x}} \right)$$

⤒ notice sign.

So the whole expression for H involves 4 major terms, call them uu, uv, vu and vv terms.

$$uu\text{-term} = \frac{1}{V} \int d^3\vec{x} \sum_{\substack{PS \\ p's'}} \sqrt{\frac{m^2}{EE'}} \overset{\boxed{E}}{\downarrow} b^*_{p's'} b_{ps} \ u(p's')^\dagger u(ps) \ e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}$$

$$= \sum_{\substack{PS \\ p's'}} \sqrt{\frac{m^2}{EE'}} \ E \ b^*_{p's'} b_{ps} \ u(p's')^\dagger u(ps) \ \delta_{\vec{p},\vec{p}'}$$

②

$$= \sum_{p\, ss'} \frac{m}{E}\, E\, b_{ps'}^{\ast}\, b_{ps}\, \underbrace{u(ps')^{+} u(ps)}_{\delta_{ss'}\frac{E}{m}}$$

since $E' = E$
$p' = P$
when $\vec{p}' = \vec{p}$

$$= \sum_{ps} E\, |b_{ps}|^{2}.$$

Sim. we find the uv and vu terms vanish, and the vv term gives

$$- \sum_{ps} E\, |c_{ps}|^{2}.$$

Thus,

$$H = \sum_{ps} E \left(|b_{ps}|^{2} - |c_{ps}|^{2} \right)$$

The calculation above is really just a check of the orthonormality of the free particle solutions in a box. The answer is obvious in the 1st quantized theory, since $|b_{ps}|^{2}$ and $|c_{ps}|^{2}$ are just the probabilities of finding $\psi(x)$ in the given (pos or neg energy) free particle states, which have energy $\pm E$. That is, $H_{cl} = \langle \psi | H_{Dirac} | \psi \rangle$.
                                                                        field

    The mode expansion above for $\psi(\vec{x})$ is taken at a fixed time, say, $t=0$. If we regard $\psi(\vec{x})$ as an initial condition for the free particle Dirac eqn, then the subsequent time evolution is easy to write down, because the plane waves in the normal modes

have a trivial time evolution. That is, if

$$\psi(\vec{x},0) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left[b_{ps}(0)\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}(0)\, v(p,s)\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

then

$$\psi(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left[b_{ps}(t)\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}(t)\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

where

$$b_{ps}(t) = b_{ps}(0)\, e^{-iEt}$$

$$c_{ps}(t) = c_{ps}(0)\, e^{+iEt}.$$

Equivalently,

$$\psi(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left[b_{ps}(0)\, u(ps)\, e^{-ip_\mu x^\mu} + c_{ps}(0)\, v(ps)\, e^{+ip_\mu x^\mu} \right].$$

The t-evolution of b_{ps} or c_{ps} (the mode amplitudes) in the complex plane is a circle (clockwise for b_{ps}, counterclockwise for c_{ps}). This looks like the evolution of a harmonic oscillator in the phase (qp) plane, and suggests that the real and imag. parts of the b_{ps} and c_{ps} are the q's and p's of the system. It is easily checked that this is correct, i.e., with this choice of q's and p's, Hamilton's (classical) eqns reproduce the ⟨correct⟩ time evolution.

This completes the classical field description of the 1st quantized Dirac theory. Now as we did with the EM field we quantize this field by

reinterpreting the q's and p's as operators satisfying the canonical commutation relations. This is Dirac's prescription for converting a classical system into a quantum one, and it is a guess that must be checked by the consistency of the quantum theory and its comparison with experiment. In the present case, it means that the mode amplitudes b_{ps}, c_{ps} become operators that satisfy the commutation relations,

$$[b_{ps}, b_{p's'}^+] = [c_{ps}, c_{p's'}^+] = \delta_{pp'} \delta_{ss'}$$

and all other commutators vanish,

$$[b_{ps}, b_{p's'}] = [b_{ps}, c_{p's'}] = [b_{ps}, c_{p's'}^+]$$

$$= [b_{ps}^+, c_{p's'}] = [b_{ps}^+, c_{p's'}^+] = [c_{ps}, c_{p's'}] = [c_{ps}^+, c_{p's'}^+] = 0.$$

Now the wave field $\psi(\vec{x})$ is a quantum field,

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{u}{E}} \left(b_{ps} \, u(ps) \, e^{i\vec{p}\cdot\vec{x}} + c_{ps} \, v(ps) \, e^{-i\vec{p}\cdot\vec{x}} \right),$$

↑ ——— now ——— ↑
operators

and the Hamiltonian H is a field operator. To obtain a field Hamiltonian with vanishing vacuum expectation value, we normal order the expression for H, which means move all b^+'s and c^+'s to the left, all b's and c's to the right, and discard any commutators.

For H this gives,

$$H = \int d^3\vec{x} \; : \; \psi^\dagger \left(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta \right) \psi :$$

$$= \sum_{ps} E \left(b_{ps}^\dagger b_{ps} - c_{ps}^\dagger c_{ps} \right).$$

where : : means, "normal order". Here there is no normal ordering to be done, since $\psi^\dagger \sim b^\dagger, c^\dagger$ is already on the left and $\psi \sim b, c$ on the right.

As yet we have little idea about what this quantum field means physically, but on the basis of our experience with photons we can expect $b_{ps}^\dagger$, $(c_{ps}^\dagger)$ to create electrons with pos (neg) energy and b_{ps}, c_{ps} to destroy them. The theory will clearly be a multiparticle theory, since we can create as many particles as we like. Recall that even the first quantized Dirac theory became a multiparticle theory when Dirac introduced his sea to overcome difficulties with the negative energy solutions.

There is one major problem with what we have done so far, however. Following photon formalism, let's create a 2-electron state with electrons in modes ps and p's'. This is either

$$b_{ps}^\dagger \, b_{p's'}^\dagger \, |0\rangle \quad \text{or} \quad b_{p's'}^\dagger \, b_{ps}^\dagger \, |0\rangle.$$

since $[b_{ps}^\dagger, b_{p's'}^\dagger] = 0$, these two states are identical, but since they differ by an exchange of electron labels and since electrons are fermions, they should differ by a sign. In other words, our field theory does not satisfy Fermi-Dirac statistics. The difficulty traces back to the assumed commutation ~~relation~~ relations of the b's and c's.

To fix this let us postulate instead that the b's and c's satisfy <u>anticommutation relations</u>, i.e.,

$$\{ b_{ps}, b_{p's'}^\dagger \} = \{ c_{ps}, c_{p's'}^\dagger \} = \delta_{pp'} \delta_{ss'}$$

all other anticommutators vanish,

$$\{ b_{ps}, b_{p's'} \} = \dots = \{ c_{ps}^\dagger, c_{p's'}^\dagger \} = 0.$$

Then we have

$$b_{ps}^\dagger b_{p's'}^\dagger |0\rangle = - b_{p's'}^\dagger b_{ps}^\dagger |0\rangle$$

and all is well with the symmetrization postulate. This does mean, however, that our b's and c's will satisfy different properties from the a's that were used in photon theory. (Of course, photons are bosons, so the commutation relations were correct for them.).

For bosons, we can put any number of particles in a mode. We have

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

⑦

where we suppress indices but one mode is referred to.

For elections, we put one election (+ energy) in mode ps with

$$b_{ps}^+ |0\rangle$$

but if we try to put two in the same mode we get

$$b_{ps}^+ \, b_{ps}^+ \, |0\rangle = 0$$

since

$$(b_{ps}^+)^2 = \underbrace{b_{ps}^+ \, b_{ps}^+}_{\text{anti commute}} = - b_{ps}^+ b_{ps}^+ = 0.$$

Similarly, we find $b_{ps}^2 = 0$. This of course is the Pauli principle: we can have either 0 or 1 election in a mode, no more. Similarly, define the number operator, (for + energy electrons)

$$N_{ps} = b_{ps}^+ b_{ps}.$$

Then

$$N_{ps}^2 = b_{ps}^+ \underbrace{b_{ps} \, b_{ps}^+}_{\text{anti comm.}} b_{ps} = - b_{ps}^+ b_{ps}^+ b_{ps} b_{ps} + b_{ps}^+ b_{ps}$$
$$= 0 + N_{ps},$$

or $\qquad N_{ps}^2 - N_{ps} = N_{ps}(N_{ps}-1) = 0.$

Thus the eigenvalues of N_{ps} are either 0 or 1.

The Hamiltonian can be written in terms of number operators,

$$H = \sum_{ps} E \left(N_{ps}^{(+)} \oplus N_{ps}^{(-)} \right).$$

where

$$N_{ps}^{(+)} = b_{ps}^{+} b_{ps} \qquad (+ \text{ energy})$$

$$N_{ps}^{(-)} = c_{ps}^{+} c_{ps} \qquad (- \text{ energy})$$

We must do some work to interpret and understand the new field theory. Let's begin by computing the Heisenberg eqns of motion for the b's and c's. This gives

$$\dot{b}_{ps} = -i \left[b_{ps}, H \right]$$

$$= -i \sum_{p's'} E' \left(\underbrace{\left[b_{ps}, b_{p's'}^{+} b_{p's'} \right]} + \underbrace{c - \text{term}} \right)$$

$$\hookrightarrow \text{vanishes.}$$

$$\hookrightarrow = b_{ps} \, b_{p's'}^{+} \, b_{p's'} - b_{p's'}^{+} \, b_{p's'} \, b_{ps}$$

$$\underset{\text{anticomm.}}{\vee}$$

$$= - b_{p's'}^{+} \, b_{ps} \, b_{p's'} - b_{p's'}^{+} \, b_{p's'} \, b_{ps} + \delta_{pp'} \delta_{ss'} \, b_{p's'}$$

$$= + b_{p's'}^{+} \, b_{p's'} \, b_{ps} - b_{p's'}^{+} \, b_{p's'} \, b_{ps} + \delta_{pp'} \delta_{ss'} \, b_{p's'}$$

so $\qquad \dot{b}_{ps} = -i E b_{ps}, \qquad$ or $\qquad b_{ps}(t) = b_{ps}(0) \, e^{-iEt}.$

Sim. we find $\qquad c_{ps}(t) = c_{ps}(0) \, e^{+iEt}.$ These are the

same eqns and solutions we found in the classical theory (now reinterpreted as operators). Thus the quantum field $\psi(\vec{r}, t)$ evolves in time by the same formula above quoted in the 1st quantized theory. (And the Dirac equ. becomes the Heisenberg equ. of motion for the quantum field ψ.)

The eigenstates of the Hamiltonian are specified by a string of occupation numbers, one for each mode, $(\cdots n_{ps}^{\pm} \cdots)$ or $\{n_{ps}^{\pm}\}$, where each $n_{ps}^{\pm} = 0$ or 1, and

$$| \cdots n_{ps}^{\pm} \cdots \rangle = \prod_{ps} (b_{ps}^{+})^{n_{ps}^{+}} \prod_{ps} (c_{ps}^{+})^{\overline{n}_{ps}} | 0 \rangle$$

where the $\pm$ on $n_{ps}^{\pm}$ means # of pos. or neg. energy electrons.

Allowing H to act on these states, we get

$$H | \cdots n_{ps}^{+} \cdots \overline{n}_{ps} \cdots \rangle = \sum_{ps} E (n_{ps}^{+} - \overline{n}_{ps}) | \cdots n_{ps}^{+} \cdots \overline{n}_{ps} \cdots \rangle.$$

For simplicity just look at a single electron state, say $b_{ps}^{+} | 0 \rangle$. Then

$$H(b_{ps}^{+} | 0 \rangle) = E(b_{ps}^{+} | 0 \rangle) \qquad \text{where } E = E(\vec{p}).$$

sim. $\quad H(c_{ps}^{+} | 0 \rangle) = -E(c_{ps}^{+} | 0 \rangle).$

The excitations have energy $E = E(\vec{p})$ (or $-E$).

What is their momentum? For this we need a field momentum

operator.   For this we go back to the 1st quantized theory
and derive a conserved momentum vector $\vec{P}$ for the Dirac field.
This follows by applying Noether's theorem to the field Lagrangian
$\mathcal{L}$, which is invariant under translations.   We find

$$\vec{P} = \int d^3\vec{x} \quad \psi^+ (-i\vec{\nabla})\psi .$$

This is simple: it is just the expectation value (in the 1st qu.
theory) of the momentum operator $-i\vec{\nabla}$.   Now we quantize
(i.e. 2nd quantize) ψ to get the field operator $\vec{P}$.   We must
normal order. We also express in terms of b's and c's. This
gives

$$\vec{P} = \int d^3\vec{x} : \psi^+ (-i\vec{\nabla})\psi : \; = \sum_{ps} \vec{p} \left(b_{ps}^+ b_{ps} - c_{ps}^+ c_{ps} \right) .$$

Thus

$$\vec{P} \left(b_{ps}^+ |0\rangle \right) = \vec{p} \left(b_{ps}^+ |0\rangle \right)$$

$$\vec{P} \left(c_{ps}^+ |0\rangle \right) = -\vec{p} \left(c_{ps}^+ |0\rangle \right).$$

The excitations $b_{ps}^+ |0\rangle$ have energy E and momentum $\vec{p}$, while $c_{ps}^+ |0\rangle$
has energy + mom. $-E$ and $-\vec{p}$.   With this we are satisfied that
the excitations should be identified with electron states.   1st qu.

We have worked $\overset{\text{so far}}{\text{with}}$ two bilinear quantities of the classical
field, H and $\vec{P}$, which we carried over to field operators. These

is another bilinear quantity important in the 1st quantized theory, namely the total probability:

$$1 = \int d^3\vec{x} \; \psi^+ \psi$$

When we (2nd) quantize ψ this becomes a field operator,

$$\int d^3\vec{x} \; :\psi^+\psi: \; = \; \sum_{ps} \left(b^+_{ps} b_{ps} + c^+_{ps} c_{ps} \right).$$

It is the sum of + and − energy number operators, so it represents the total # of electrons in the system (either pos or neg. energy). If we multiply by $q = -e$ we get a __charge operator__,

$$Q = -e \int d^3\vec{x} \; :\psi^+\psi: \; = \; -e \sum_{ps} \left(b^+_{ps} b_{ps} + c^+_{ps} c_{ps} \right)$$

At this point we have a 2nd quantized version of Dirac's theory of pos and neg. energy electrons. It gives the correct FD statistics in multiparticle problems, but otherwise its physical content does not go beyond what we had with the first quantized theory. In particular, it does not address the interpretational difficulties of the neg. energy solutions.

To fix this up, we borrow ideas from hole theory. If all the negative energy states are filled, as Dirac supposed, then when we

⑫

excite an electron out of a negative energy state, we create a hole. Thus, the <u>destruction</u> of a neg. energy electron in the sea is equivalent to the <u>creation</u> of a hole (or positron). Thus let us define

$$c_{ps} = d^{\dagger}_{ps}$$

where $d^{\dagger}_{ps}$ creates a position of charge, energy, momentum and spin $+e, +E, +\vec{p}$ and $+S$. Likewise, if an electron makes a radiative transition to an unoccupied neg. energy state (a hole), it <u>creates</u> a neg. energy electron or <u>destroys</u> a hole. So let us write

$$c^{\dagger}_{ps} = d_{ps}.$$

Note that with these defn's, the $d_{ps}, d^{\dagger}_{ps}$ satisfy anticomm-relations exactly like the b's and c's,

$$\{d_{ps}, d^{\dagger}_{p's'}\} = \delta_{pp'}\delta_{ss'}$$

all other $\{\,,\,\} = 0$.

Now the quantum field is

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{ps} \sqrt{\frac{m}{E}} \left(b_{ps}\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + d^{\dagger}_{ps}\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}} \right).$$

also, the field Hamiltonian becomes,

$$H = \sum_{ps} E \left(b_{ps}^{+} b_{ps} - d_{ps} d_{ps}^{+} \right).$$

Notice the ordering of the d operators. If we anticommute them, we get

$$H = \sum_{ps} E \left(b_{ps}^{+} b_{ps} + d_{ps}^{+} d_{ps} \right) - \sum_{ps} E$$

The final term is ∞. One interpretation is that it is the ∞ (neg.) energy of the filled Dirac sea. Another interp. is that it is an ∞ term resulting from the ordering ambiguities on passing from the classical expression to a qu. operator. That is, it is a zero point term.

We throw away zero point terms to make vacuum expectation values vanish. But is the vacuum the state without any electrons of any kind, pos or neg. energy, or is it the state without either electrons or positrons? The latter is more physical, and if we want $\langle 0|H|0 \rangle = 0$ for this vacuum, then we must throw away the term $-\sum_{ps} E$ above.

This leads to a new interpretation of normal ordering:
We migrate b^{+}'s and d^{+}'s to the left, b's and d's to the right, keeping any sign changes on applying anticommutators, but

throwing away the anticommutators themselves. This differs from what we did above in that we use $d^\dagger, d$ not $c^\dagger, c$. Thus,

$$H = \int d^3\vec{x} \, :\psi^\dagger \left(-i\vec{\alpha}\cdot\nabla + m\beta\right)\psi: \; = \; \sum_{PS} E \left(b^\dagger_{PS} b_{PS} + d^\dagger_{PS} d_{PS}\right)$$

$$\vec{P} = \int d^3\vec{x} \, :\psi^\dagger \left(-i\nabla\right)\psi: \; = \; \sum_{PS} \vec{P} \left(b^\dagger_{PS} b_{PS} + d^\dagger_{PS} d_{PS}\right)$$

$$Q = \int d^3\vec{x} \, :\psi^\dagger\psi: \; = \; -e\sum_{PS} \left(b^\dagger_{PS} b_{PS} - d^\dagger_{PS} d_{PS}\right).$$

Notice the sign changes. Now positrons have positive energy, momentum (and spin), but the charge operator is no longer positive definite. The quantity Dirac worked so hard to make positive definite in the 1st quantized theory is now replaced by an operator of either sign in the 2nd quantized theory. Energies are strictly positive, and we deal strictly with observable objects (electrons and positrons). Dirac's original goal of "curing" the "problems" with the KG eqn now appears as irrelevant, although there was no way to see that within the framework of the single particle (1st quantized) theory. The real conceptual breakthrough came with hole theory.

Now consider the Dirac electron field (2nd quantized) interacting with the EM field. Initially for simplicity we take the EM field as a specified $\overset{\text{given}}{\vee}$ c-number field $A_\mu(\vec{x}, t)$. So we don't need to worry about photons or Maxwell eqns.

To obtain the form of interaction, we go back to the free Dirac Lagrangian and use ~~the~~ the minimal coupling prescription,

$$\mathcal{L} = \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi, \, = \mathcal{L}_{free} \quad \text{(free particle)}$$

$$i\partial_\mu \longrightarrow i\partial_\mu \ominus q A_\mu,$$

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int},$$

$$\mathcal{L}_{int} = -q\bar{\psi}\gamma^\mu A_\mu \psi = -q\bar{\psi}\not{A}\psi = -A_\mu J^\mu$$

where $J_\mu = q\bar{\psi}\gamma^\mu\psi$ is the current. Then for the Hamiltonian density we get

$$\mathcal{H} = \pi\dot{\psi} + \bar{\pi}\dot{\bar{\psi}} - \mathcal{L}$$
$$= \mathcal{H}_{free} + \mathcal{H}_{int},$$

$$\mathcal{H}_{free} = \psi^\dagger(-i\vec{\alpha}\cdot\nabla + m\beta)\psi$$

$$\mathcal{H}_{int} = -\mathcal{L}_{int} = q\bar{\psi}\gamma^\mu A_\mu \psi.$$

(Notice $\mathcal{L}_{int}$ is a Lorentz scalar, as required for a Lorentz covariant theory.)

This is at the classical level. To quantize we reinterpret everything as operators and normal order. Thus the Hamiltonian is

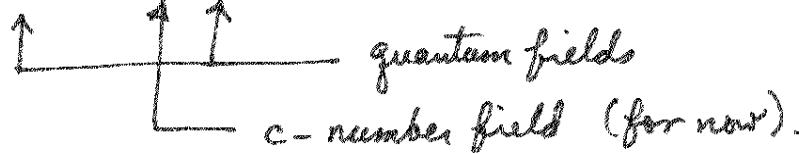$$H = \int d^3\vec{x} \; \mathcal{H} \;=\; H_{free} + H_{int} \;=\; H_0 + H_1,$$

where

electrons   positrons
↓   ↓

$$H_0 = \int d^3\vec{x} \; : \psi^\dagger \left(-i\vec{\alpha}\cdot\nabla + m\beta\right)\psi : \;=\; \sum_{PS} E\left(b^\dagger_{PS} b_{PS} + d^\dagger_{PS} d_{PS}\right)$$

$$H_1 = q\int d^3\vec{x} \; : \bar{\psi}\, \gamma^\mu A_\mu \psi :$$

⎽⎼⎽⎽⎼⎯⎯ quantum fields
⎽⎼⎯ c-number field (for now).

Let's begin by looking at the processes that can be engendered by H_1 in 1st order TDPT. The matrix element is $\langle f|H_1|i\rangle$, and it has the general structure,

$$\langle f|H_1|i\rangle \sim \int d^3\vec{x} \; \langle f| : \sum_{PS}\left(\cdots b^\dagger_{PS}\cdots d_{PS}\right) \gamma^\mu A_\mu(\vec{x},t) \sum_{p's'}\left(b_{p's'}\cdots d^\dagger_{p's'}\right):|i\rangle$$

omitting all normalization consts etc. There are 4 types of terms that are formed from creation/annihil. operators, ⚌ as shown in the table: