Second quantizing the Dirac equation means reinterpreting the wave function ψ, which is supposed to represent the QM of a single particle (this is the <u>first</u> quantized version of the theory) as a <u>quantum field</u>. Thus, ψ goes from a c-number field into an operator field. The resulting quantum field theory is capable of handling problems with multiple electrons and positrons, including processes in which particles are created or destroyed.

To second quantize ψ, we follow the same general outline used earlier for quantizing the electromagnetic field. First we identify the classical field Hamiltonian (thinking of the first quantized ψ as a "classical" field) and the set of q's and p's that reproduce the desired field equations via classical Hamilton's equations. In the case of the 1st quantized Dirac field, these field equations are the Dirac equation,

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\nabla\psi + \beta\psi \qquad (m=\hbar=1).$$

or $$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

Then we replace the classical q's and p's with <u>operators</u>, obtaining a quantum field for ψ. The excitations of the field (the energy eigenstates) are then identified with particles. This follows what we did for photons.

To find the field Hamiltonian for the first quantized Dirac field ψ, we first find a Lagrangian. In field theory, the Lagrangian L is a spatial integral of a Lagrangian density $\mathcal{L}$,

$$L = \int d^3\vec{x} \; \mathcal{L}$$

where the integral represents the sum over the degrees of freedom of the field (one for each point of space). In simple applications, $\mathcal{L}$ is a function of the field (call it ψ in a general notation) and its first derivatives, $\mathcal{L} = \mathcal{L}(\psi, \partial_\mu \psi)$. This is like the Lagrangians $L(q, \dot{q})$ in ordinary particle mechanics. The action S is the time-integral of the Lagrangian,

$$S = \int dt \; L = \int d^4x \; \mathcal{L},$$

and by Hamilton's principle the equations of motion are equivalent to $\delta S = 0$ (S is stationary w.r.t. variations in the field ψ). This implies the Euler-Lagrange equations for the field,

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) = \frac{\partial \mathcal{L}}{\partial \psi}.$$

This is for a single field ψ; if there is more than one field (or if ψ has multiple components), then there is a separate

Euler - Lagrange eqn. for each field / component,

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\right) = \frac{\partial \mathcal{L}}{\partial \psi_a}, \qquad a = 1, 2, \ldots$$

Compare the E-L eqns in ordinary particle mechanics,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}, \qquad i = 1, 2, \ldots$$

If we wish the equations of motion to be relativistically covariant, then the action S must be a Lorentz scalar. Since the space-time volume element d^4x is a Lorentz scalar, this implies that the Lagrangian density $\mathcal{L}$ must be a Lorentz scalar, too. This imposes severe constraints on the form $\mathcal{L}$ is allowed to take in a covariant theory.

For example, for the free electromagnetic field, the simplest scalars that can be constructed from the field A^μ and its first derivatives are $F_{\mu\nu}F^{\mu\nu}$ and $A_\mu A^\mu$, of which only the first is gauge invariant. As it turns out, the $\mathcal{L}$ for the free EM field is

$$\underset{\sim}{\mathcal{L}} = \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$$

where the $1/16\pi$ is conventional. If you add a term proportional to $A^\mu A_\mu$, it gives the photon ~~an~~ a nonzero mass (and breaks gauge invariance).

For the free Dirac (first quantized or c-number) field, the simplest Lorentz scalars we can construct out of (the 4-component) ψ are $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\partial_\mu\psi$. By taking linear combinations of these we find a $\mathcal{L}$ that works:

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \qquad (\hbar = c = 1).$$

We can check that this works. There are really 8 (real) fields, the real and imag parts of ψ_a, $a = 1,2,3,4$. Equivalently, we can treat ψ and ψ^+, or ψ and $\bar{\psi} = \psi^+\gamma^0$, as independent fields. The required derivatives are

$$\frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}i\gamma^\mu$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0.$$

The $\bar{\psi}$ equation immediately gives the Dirac eqn,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

The ψ equation does, too, in adjoint form:

$$\partial_\mu \left(\bar{\psi} i \gamma^\mu \right) = - m \bar{\psi}$$

$$\Rightarrow \quad (\partial_\mu \bar{\psi}) i \gamma^\mu = i (\partial_\mu \psi^\dagger) \gamma^0 \gamma^\mu = - m \psi^\dagger \gamma^0$$

$$\Rightarrow \quad i (\partial_\mu \psi^\dagger) \gamma^0 \gamma^\mu \gamma^0 + m \psi^\dagger = 0$$

$$\Rightarrow \quad -i \, \gamma^0 (\gamma^\mu)^\dagger \gamma^0 \, \partial_\mu \psi + m \psi = 0$$

$$\Rightarrow \quad i \gamma^\mu \partial_\mu \psi - m \psi = 0.$$

So $\mathcal{L}_D$ is an acceptable field Lagrangian for the free Dirac field.

Now we need the field Hamiltonian. In ordinary particle mechanics we define the canonical momentum by

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

and then the Hamiltonian by

$$H = \sum_i p_i \dot{q}_i - L.$$

In field theory, we define the <u>momentum fields</u> π_a by

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a}, \quad \text{where} \quad \dot{\psi}_a = \frac{\partial \psi_a}{\partial t},$$

and then the Hamiltonian density by

$$\mathcal{H} = \sum_a \pi_a \dot{\psi}_a - \mathcal{L} .$$

Finally, the Hamiltonian is

$$H = \int d^3\vec{x} \;\; \mathcal{H} .$$

To apply this to the Dirac field, write the Lagrangian as

$$\mathcal{L}_D = \bar{\psi} \left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot\nabla - m \right) \psi .$$

Then

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0$$

$$\bar{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

so

$$\mathcal{H} = \pi\dot{\psi} + \bar{\pi}\dot{\bar{\psi}} - \mathcal{L} = i\bar{\psi}\gamma^0 \frac{\partial \psi}{\partial t} - \mathcal{L}$$

$$= \bar{\psi}\left(-i\vec{\gamma}\cdot\nabla + m \right)\psi$$

$$= \psi^+ \left(-i\vec{\alpha}\cdot\nabla + m\beta \right)\psi .$$

The Hamiltonian density is $\psi^+ \psi$ sandwiched around the usual Dirac (quantum) Hamiltonian operator. Thus, the field Hamiltonian is

$$H = \int d^3\vec{x} \;\; \psi^+ \left(-i\vec{\alpha}\cdot\nabla + m\beta \right)\psi$$

It is the expec. val. of the qu. Hamiltonian w.r.t. the state ψ.

$\hbar = c = 1$ in the following.

Compare the field Hamiltonian for the (1st quantized) Dirac field with that of the EM field:

$$H = \int d^3\vec{x}\ \psi^\dagger(-i\vec{\alpha}\cdot\nabla + m\beta)\psi \qquad \text{vs.} \qquad H = \int d^3\vec{x}\ \frac{E^2+B^2}{8\pi}.$$

These are classical Hamiltonians in the sense that when the right q's and p's are identified, then the correct eqns. of motion follow from the classical Hamilton's eqns. In the case of the Dirac eqn, the correct eqns of motion are

$$i\frac{\partial\psi}{\partial t} = (-i\vec{\alpha}\cdot\nabla + m\beta)\psi$$

and in the case of the EM field they are Maxwell's eqns. The q's and p's can be found systematically by proceeding from ~~a~~ the Lagrangian, but it's easier just to guess them (and then check the answers). We do this by developing the normal mode expansion of the Dirac (1st quantized) field.

The normal mode expansion of the classical EM field is a representation of $\vec{A}$ as a lin. comb. of plane light waves,

$$\vec{A}(\vec{x}) = \sqrt{\frac{2\pi}{V}} \sum_{\vec{k}\mu} \frac{1}{\sqrt{\omega}}\left(a_{\vec{k}\mu}\, \hat{\epsilon}_{\vec{k}\mu}\, e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}\right).$$

For the (1st quantized) Dirac field, it is a representation of ~~[illegible]~~ $\psi(\vec{x})$ (the 4-comp. spinor) as a lin. comb. of free particle (plane wave) solutions. These are

$$u(p,s)\, e^{i\vec{p}\cdot\vec{x}} \qquad \text{(pos. energy)}$$

and $$v(p,s)\, e^{-i\vec{p}\cdot\vec{x}} \qquad \text{(neg. energy)}.$$

These are parameterized by a 3-momentum $\vec{p}$, we define $E=\sqrt{m^2+|\vec{p}|^2}$ (the pos. square root) and $p=(E,\vec{p})$ the 4-vector. Both E, p are determined once $\vec{p}$ is given. s is the spin 4-vector which takes on only 2 values ($\pm$ some direction in the rest frame). The energy and momentum eigenvalues of the pos. energy solns are $E, \vec{p}$, and of the neg. energy solns, $-E, -\vec{p}$. For given $\vec{p}$ there are 4 solns ($\pm s$, u and v). The spinors u, v satisfy the Hermitian orthonormality conditions,

$$u(p,s)^\dagger u(p,s') = \frac{E}{m}\delta_{ss'}$$

$$u(p,s)^\dagger v(\tilde{p},s') = 0$$

$$v(p,s)^\dagger u(\tilde{p},s') = 0$$

$$v(p,s)^\dagger v(p,s') = \frac{E}{m}\delta_{ss'}$$

where $\tilde{p}=(E,-\vec{p})$.

We normalize the free particle solns above in a box w. vol. $V=L^3$:

Dirac's prob. density.

$$\int_{\text{box}} d^3\vec{x}\ \psi^\dagger\psi = 1$$

This means

integer vector

$$\vec{p} = \frac{2\pi}{L}\vec{n} = \underline{\text{discrete}}$$

Thus, including normalization, the free particle solutions are

$$\frac{1}{\sqrt{V}}\sqrt{\frac{m}{E}}\; u(p,s)\; e^{i\vec{p}\cdot\vec{x}}$$

$$\frac{1}{\sqrt{V}}\sqrt{\frac{m}{E}}\; v(p,s)\; e^{-i\vec{p}\cdot\vec{x}}$$

An arbitrary Dirac wave fn ψ can be represented as a lin. comb. of these solutions,

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}}\sum_{ps}\sqrt{\frac{m}{E}}\left(b_{ps}\, u(ps)\, e^{i\vec{p}\cdot\vec{x}} + c_{ps}\, v(ps)\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

where b_{ps}, c_{ps} are the expansion coefficients. This is because the free particle Dirac Hamiltonian is Hermitian and complete. The p in the $\sum_p$ means really $\sum_{\vec{p}}$, and $\sum_s$ means sum over 2 values of s corresp. to $\pm\hat{s}$ in the rest frame. We call b_{ps} and c_{ps} the <u>mode</u> <u>amplitudes</u>. They are (in the 1st quantized Dirac theory) the probability amplitudes to find the state ψ in a given free particle state of + or − energy with mode ps $(=\vec{p},s)$.

Let us now express the field Hamiltonian in terms of the mode amplitudes. Recall when we did this for the free EM field, we obtained

$$H = \int d^3\vec{x}\;\frac{E^2+B^2}{8\pi} = \sum_{\vec{k}\mu}\hbar\omega\;|a_{\vec{k}\mu}|^2 .$$