These are some notes on the Foldy-Wouthuysen (F-W) transformation. They are not intended to be complete, just to bridge the gap between my lectures and Ch. 4 of Bjorken and Drell.

First, I was more systematic about the (v/c) ordering of various terms than B+D. As in B+D, we use <u>natural</u> <u>units</u>, where $\hbar = c = 1$. These are different from atomic units, where $e = m = \hbar = 1$ (m = electron mass). In natural units, the fine structure constant becomes

$$\alpha = \frac{e^2}{\hbar c} \rightarrow e^2 = \frac{1}{137}.$$

Let a_0 = Bohr radius = unit of dist in atomic units. The unit of distance (for an electron) in natural units is the Compton wavelength,

$$\lambda_c = \frac{\hbar}{mc} \rightarrow \frac{1}{m} \text{ (natural units)}.$$

Since $a_0 = \frac{\lambda_c}{\alpha} \approx 137 \lambda_c$ (in any units), we have $a_0 = \frac{1}{\alpha m}$ in natural units. Similarly, let τ_0 = unit of time in atomic units, $\tau_0 = \frac{a_0}{\alpha c}$ ~ orbital period of electron in H-atom. The unit of time in natural units is $\frac{\hbar}{mc^2} = \alpha^2 \tau_0$. So $\tau_0 = \frac{1}{\alpha^2 m}$ in natural units.

Now we wish to order terms in powers of v/c. Introduce a formal small parameter η where $\eta = O(v/c)$. The only purpose of η is to keep track of the order of various terms.

The ordering depends on the physical situation. Suppose we take the ground state of hydrogen as a reference. As an order of magnitude, the same estimates would apply to any atom near the ground state with low Z. We assume that all $\vec{E}$ and $\vec{B}$ fields are internally generated.

The electron velocity is $v \sim \alpha c$, or, in natural units, $v \sim \alpha$. But $\frac{v}{c} = v \sim \eta^1$, so $\alpha \sim \eta^1$. (First power, $\eta^1 = \eta$). We take the electron mass to be $m \sim \eta^0$ (independent of η). The kinetic energy is $\frac{1}{2} m v^2 \sim \eta^2$. Since kinetic and potential energy trade off during the motion of the electron, the potential energy $q\Phi \sim \eta^2$, too. So far:

Qty	Order in η
m	η^0
v	η
$\frac{1}{2} m v^2$	η^2
$q\Phi$	η^2

Here is another way to see that $q\Phi$ is of order η^2. The Coulomb potential Φ in an atom goes like $\frac{q}{a_0}$ so $q\Phi$ goes like $\frac{q^2}{a_0} = \frac{e^2}{a_0} = \frac{\alpha}{a_0} = m\alpha^2 \sim \eta^2$ (working in natural units). Here we use $a_0 = \frac{1}{m\alpha}$. Add to the table:

Qty	order in η
a_0	η^{-1}
τ_0	η^{-2}

The Dirac matrices $\vec{\alpha}, \beta, \gamma^\mu, \gamma_5, \vec{\Sigma}, \sigma^{\mu\nu}$, etc. are all of order unity (they are dimensionless).

As we analyzed Φ we can also analyze $\vec{A}$. As an order of magnitude $\vec{A} \sim \frac{v}{c}\Phi$ (Gaussian units) where v is the velocity of the source charge. This means $q\vec{A}$ is one order of $\eta \sim \frac{v}{c}$ higher than $q\Phi$, or,

$$q\vec{A} \sim \eta^3 .$$

The fields $\vec{A}, \Phi, \vec{E}, \vec{B}$ etc vary on a scale length of a_0, so the operator ∇ goes like $\frac{1}{a_0}$ which is order η. Similarly, $\partial/\partial t$ goes like $1/\tau_0$ which is order η^2. Thus:

$$\nabla \sim \eta$$

$$\frac{\partial}{\partial t} \sim \eta^2 .$$

This means that

$$q\vec{B} = q(\nabla \times \vec{A}) \sim \eta^4$$

and

$\nabla\Phi$ dominates.

$$q\vec{E} = -q\nabla\Phi - q\frac{\partial \vec{A}}{\partial t} \sim \eta^3 + \eta^5$$

To summarize,

Qty	order in η
∇	η
$\frac{\partial}{\partial t}$	η^2
$q\vec{A}$	η^3
$q\vec{B}$	η^4
$q\vec{E}$	η^3

In other physical circumstances you will have different ordering schemes.

Now following B+D, write the Dirac equation in natural units as

$$i\frac{\partial\psi}{\partial t} = H\psi$$

obviously don't confuse the Dirac matrices $\vec{\alpha}$ with the fine structure const. α

where

$$H = \underset{\eta^0}{m\beta} + \underset{\eta^1}{\vec{\alpha}\cdot\vec{\pi}} + \underset{\eta^2}{q\Phi}$$

where the order of the terms is given. Here $\vec{\pi} = \vec{p} - q\vec{A}$ $\sim\eta$ because $\vec{p}\sim m\vec{v}\sim\eta$ (and the $q\vec{A}$ term is η^3, much smaller). We organize Dirac matrices as even or odd as in B+D, and write

$$\mathcal{O} = \vec{\alpha}\cdot\vec{\pi}, \qquad \mathcal{E} = q\Phi,$$

so
$$\overset{\eta^0}{m\beta} + \overset{\eta^1}{\mathcal{O}} + \overset{\eta^2}{\mathcal{E}}$$
$$H = m\beta + \mathcal{O} + \mathcal{E}.$$

The aim is to transform ψ and H by a unitary transformation that will decouple the upper and lower 2-component spinors. We work in the Dirac-Pauli representation of the Dirac matrices, which is most convenient for the nonrelativistic limit. We achieve the decoupling if we can transform away the odd Dirac matrices.

We write the transformation as
$$\psi' = e^S \psi.$$

B+D use iS where I use S; I have absorbed the i into S because it simplifies the algebra. Thus, my S must be anti-Hermitian so that e^S is unitary. Substituting this into the Dirac equation, we get

$$i \frac{\partial \psi'}{\partial t} = H' \psi'$$

where
$$H' = e^S H e^{-S} + i \left(\frac{\partial e^S}{\partial t} \right) e^{-S}.$$

We assume S is small (of order η or higher), since H is already even at order η^0 (the term $m\beta$), and the first odd term we have to get rid of is $\mathcal{O}$ (at order η'). This means e^S, e^{-S} etc can be expanded in power series in S.

This expansion was done in HW 1 from the fall semester (see problem 1.2 of the notes, and the solutions). As in that problem, we write $[S,X] = L_S X$, $[S,[S,X]] = L_S^2 X$, etc., for any operator X. Then we have

$$e^S H e^{-S} = H + L_S H + \tfrac{1}{2} L_S^2 H + \tfrac{1}{6} L_S^3 H + \tfrac{1}{24} L_S^4 H + \dots$$

and
$$\left(\frac{\partial}{\partial t} e^S\right) e^{-S} = \dot{S} + \tfrac{1}{2} L_S \dot{S} + \tfrac{1}{6} L_S^2 \dot{S} + \dots$$

where $\dot{S} = \frac{\partial S}{\partial t}$. We are allowing $\Phi, \vec{A}$ in the Dirac equation to depend on time, so we must allow S to depend on time also.

To start, let's choose S to transform away the odd terms in H through order η. This is the term $\mathcal{O} = \vec{\alpha}\cdot\vec{\pi}$. Through $O(\eta^1)$ we have

$$H = m\beta + \mathcal{O} + \dots$$

and
$$\begin{aligned} H' = {} & m\beta + L_S(m\beta) + \tfrac{1}{2} L_S^2 (m\beta) + \dots \\ & + \mathcal{O} + L_S(\mathcal{O}) + \tfrac{1}{2} L_S^2(\mathcal{O}) + \dots \\ & + i\left(\dot{S} + L_S \dot{S} + \dots\right) \end{aligned}$$

We'll choose S so that $L_S(m\beta) = m[S,\beta]$ cancels $\mathcal{O}$, that is,

$$L_S(m\beta) = -\mathcal{O}.$$

This makes $S = O(\eta^1)$ and $\dot{S} = O(\eta^3)$ at least, so all other terms are order η^2 or higher. So we have,

$$H = m\beta + \mathcal{O} + \cdots$$

$$H' = m\beta \qquad + \cdots$$

(neglected terms are $\mathcal{O}(\eta^2)$ or higher)

So we need to solve ~~for~~

$$L_S(m\beta) = -\mathcal{O} = [S, m\beta]$$

for S. Here

$$\mathcal{O} = \vec{\alpha}\cdot\vec{\pi} = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\pi} \\ \vec{\sigma}\cdot\vec{\pi} & 0 \end{pmatrix} \text{ as a Dirac matrix.}$$

Let

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with as-yet-unknown entries a b c d.

Then

$$[S, m\beta] = m[S,\beta] = m\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$

$$= m\begin{pmatrix} 0 & -2b \\ 2c & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\pi} \\ \vec{\sigma}\cdot\vec{\pi} & 0 \end{pmatrix},$$

or

$$b = -c = \frac{1}{2m}\vec{\sigma}\cdot\vec{\pi}, \quad \text{and}$$

$$S = \frac{1}{2m}\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\pi} \\ -\vec{\sigma}\cdot\vec{\pi} & 0 \end{pmatrix} = \frac{1}{2m}\beta\,\vec{\alpha}\cdot\vec{\pi} = \frac{1}{2m}\beta\,\mathcal{O}.$$

We see that $S \sim \eta'$, as expected. Now write out H' to fourth order in η. As for $\dot{S}$, it is

$$\dot{S} = \frac{1}{2m}\beta\vec{\alpha}\cdot\frac{\partial\vec{\pi}}{\partial t} = -\frac{q}{2m}\beta\vec{\alpha}\cdot\frac{\partial\vec{A}}{\partial t} \sim \eta^5.$$

Thus through 4-th order, the terms with $\dot{S}$ don't even appear. So we get:

$$\begin{array}{llllll} & \eta^0 & \eta^1 & \eta^2 & \eta^3 & \eta^4 \\ H' = & m\beta & + L_S(m\beta) & + \frac{1}{2}L_S^2(m\beta) & + \frac{1}{6}L_S^3(m\beta) & + \frac{1}{24}L_S^4(m\beta) + \ldots \\ & & + \mathcal{O} & + L_S\mathcal{O} & + \frac{1}{2}L_S^2\mathcal{O} & + \frac{1}{6}L_S^3\mathcal{O} + \ldots \\ & & & + \mathcal{E} & + L_S\mathcal{E} & + \frac{1}{2}L_S^2\mathcal{E} + \ldots \end{array}$$

Now using $L_S(m\beta) = -\mathcal{O}$, the column η' cancels (that was the design). Also, $L_S^2(m\beta) = -L_S\mathcal{O}$, so in the (η^2) column we get $-\frac{1}{2}L_S\mathcal{O} + L_S\mathcal{O} = \frac{1}{2}L_S\mathcal{O}$; in the η^3 column, $-\frac{1}{6}L_S^2\mathcal{O} + \frac{1}{2}L_S^2\mathcal{O} = \frac{1}{3}L_S^2\mathcal{O}$; and in the η^4 column, $-\frac{1}{24}L_S^3\mathcal{O} + \frac{1}{6}L_S^3\mathcal{O} = \frac{1}{8}L_S^3\mathcal{O}$. So,

$$\begin{array}{lllll} & \eta^0 & \eta^2 & \eta^3 & \eta^4 \\ H' = & m\beta & + \frac{1}{2}L_S\mathcal{O} & + \frac{1}{3}L_S^2\mathcal{O} & + \frac{1}{8}L_S^3\mathcal{O} + \ldots \\ & & + \mathcal{E} & + L_S\mathcal{E} & + \frac{1}{2}L_S^2\mathcal{E} + \ldots \end{array}$$

Obviously we need the commutators $L_S\mathcal{O} = [S,\mathcal{O}]$, $L_S^2\mathcal{O} = [S,[S,\mathcal{O}]]$, etc. These are easily worked out, for example,

$$L_S \theta = [S, \theta] = \left[\frac{1}{2m}\beta\theta, \theta\right] = \frac{1}{2m}\left(\beta\theta^2 - \theta\beta\theta\right)$$

$$= \frac{1}{m}\beta\theta^2$$

where we use $\quad \{\beta, \theta\} = \{\beta, \vec{\alpha}\cdot\vec{\pi}\} = 0, \quad \beta\theta = -\theta\beta.$

Similarly,

$$L_S^2\theta = [S, L_S\theta] = \left[\frac{1}{2m}\beta\theta, \frac{1}{m}\beta\theta^2\right]$$

$$= \frac{1}{2m^2}\left(\beta\theta\beta\theta^2 - \beta\theta^2\beta\theta\right) = \frac{-1}{m^2}\theta^3$$

again using $\{\beta, \theta\} = 0$ and $\beta^2 = 1$. Finally,

$$L_S^3\theta = [S, L_S^2\theta] = \left[\frac{1}{2m}\beta\theta, -\frac{1}{m^2}\theta^3\right] = -\frac{1}{2m^3}\left(\beta\theta^4 - \theta^3\beta\theta\right)$$

$$= -\frac{1}{m^3}\beta\theta^4.$$

~~We also need the commutators $L_S\mathcal{E} = [S, \mathcal{E}]$ and $L_S^2\mathcal{E} = [S, [S, \mathcal{E}]]$.~~ not yet

Let's look at what we have now through order η^2:

$$H' = m\beta + 0 + H_2',$$

where $H_2' = \frac{1}{2}L_S\theta + \mathcal{E} = \frac{1}{2m}\beta\theta^2 + q\Phi.$

Now $\theta^2 = (\vec{\alpha}\cdot\vec{\pi})^2 = \alpha_i\pi_i\alpha_j\pi_j = \alpha_i\alpha_j\pi_i\pi_j,$

because $\vec{\alpha}$ is a pure spin operator and $\vec{\pi}$ is purely spatial.

Now it's easy to verify the useful identity,

$$\alpha_i \alpha_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\Sigma_k,$$

so that

$$(\vec{\alpha}\cdot\vec{\pi})^2 = \pi^2 + \underbrace{i\,\varepsilon_{ijk}\,\Sigma_k\,\pi_i\pi_j}.$$

$$\hookrightarrow = \frac{i}{2}\,\varepsilon_{ijk}\,\Sigma_k\,\underbrace{(\pi_i\pi_j - \pi_j\pi_i)}$$

$$\hookrightarrow [\pi_i, \pi_j] = iq\,\varepsilon_{ijl}\,B_l.$$

Use $\varepsilon_{ijk}\,\varepsilon_{ijl} = 2\delta_{kl}$ and the last term becomes,

$$(\vec{\alpha}\cdot\vec{\pi})^2 = \pi^2 - q\,\vec{\Sigma}\cdot\vec{B}.$$

altogether, we have (through order η^2),

$$H' = \beta\left(m + \frac{1}{2m}\pi^2 - \frac{q}{2m}\vec{\Sigma}\cdot\vec{B}\right) + q\Phi$$

$$= \beta\left(m + \frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\Sigma}\cdot\vec{B}\right) + q\Phi.$$

H' is block diagonal through this order, and if we just look at the upper 2×2 block, ~~where~~ we must replace $\beta \to 1$ and $\vec{\Sigma} \to \vec{\sigma}$. Then we get

$$H' = m + \frac{1}{2m}(\vec{p} - q\vec{A}) - \frac{q}{2m}\vec{\sigma}\cdot\vec{B} + q\Phi.$$

This is the correct Pauli Hamiltonian, including the $-\vec{\mu}\cdot\vec{B}$ term with the correct electron g-factor, $g=2$, expressed in natural units with the rest mass $m \to mc^2$ added.

To go to higher order, we must get H_3' and H_4', where

$$H_3' = \frac{1}{3} L_S^2 \mathcal{O} + L_S \mathcal{E}$$

$$H_4' = \frac{1}{8} L_S^3 \mathcal{O} + \frac{1}{2} L_S^2 \mathcal{E}.$$

As for H_3', we already found $L_S^2 \mathcal{O}$, and

$$\frac{1}{3} L_S^2 \mathcal{O} = -\frac{1}{3m^2} \mathcal{O}^3 .$$

As for $L_S \mathcal{E}$, it is

$$L_S \mathcal{E} = [S, q\Phi] = \frac{q}{2m}[\beta \vec{\alpha}\cdot\vec{\pi}, \Phi] .$$

We can replace $\vec{\pi}$ by $\vec{p}$ since this term is already at 3rd order and the correction $-q\vec{A}$ will be 2 orders higher, 5th order, which is off the charts. So,

$$L_S \mathcal{E} = \frac{q}{2m} \beta \vec{\alpha} \cdot [\vec{p}, \Phi]$$

where we remove the spin operators $\beta\vec{\alpha}$ from the commutator since they commute with the purely spatial operator Φ. But $[\vec{p}, \Phi] = -i\nabla\Phi = i\vec{E}$ where we drop the term in $\frac{\partial \vec{A}}{\partial t}$ since its order is off the charts. So, $L_S \mathcal{E} = \frac{iq}{2m} \beta \vec{\alpha}\cdot\vec{E}$.

altogether,

$$H_3' = \frac{-1}{3m^2}\mathcal{O}^3 + \frac{iq}{4m}\beta\vec{\alpha}\cdot\vec{E}.$$

It is odd.

Now for H_4'. We already have $L_S^3\mathcal{O} = -\frac{1}{m^3}\beta\mathcal{O}^4$,

so

$$\frac{1}{8}L_S^3\mathcal{O} = -\frac{1}{8m^3}\beta\mathcal{O}^4 = -\frac{1}{8m^3}\beta\left[(\vec{\alpha}\cdot\vec{\pi})^2\right]^2$$

$$= -\frac{1}{8m^3}\beta\left[\pi^2 - q\vec{\Sigma}\cdot\vec{B}\right]^2 \longrightarrow \frac{-1}{8m^3}\beta\pi^2$$

where we drop terms in $\vec{\Sigma}\cdot\vec{B}$ because they are beyond 4th order. For the same reason we drop the $q\vec{A}$ correction in π, so this term becomes

$$\frac{1}{8}L_S^3\mathcal{O} = -\frac{1}{8m^3}\beta p^4 + \text{higher order}$$

As for $\frac{1}{2}L_S^2\mathcal{E}$, we have

$$\frac{1}{2}L_S^2\mathcal{E} = \frac{1}{2}[S, L_S\mathcal{E}] = \frac{1}{2}\left[\frac{1}{2m}\beta(\vec{\alpha}\cdot\vec{\pi}), \frac{iq}{2m}\beta(\vec{\alpha}\cdot\vec{E})\right]$$

$$= \frac{iq}{8m^2}\left[\beta(\vec{\alpha}\cdot\vec{\pi}), \beta(\vec{\alpha}\cdot\vec{E})\right]$$

$$= \frac{iq}{8m^2}\left[\beta(\vec{\alpha}\cdot\vec{\pi})\beta(\vec{\alpha}\cdot\vec{E}) - \beta(\vec{\alpha}\cdot\vec{E})\beta(\vec{\alpha}\cdot\vec{\pi})\right]$$

$$= -\frac{iq}{8m^2}\left[(\vec{\alpha}\cdot\vec{\pi})(\vec{\alpha}\cdot\vec{E}) - (\vec{\alpha}\cdot\vec{E})(\vec{\alpha}\cdot\vec{\pi})\right]$$

where we anti-commute β past $\vec{\alpha}$ and use $\beta^2=1$. Thus

$$\tfrac{1}{2}L_s^2\,\mathcal{E} = -\frac{iq}{8m^2}\left[\alpha_i\pi_i\,\alpha_j E_j - \alpha_i E_i\,\alpha_j\pi_j\right]$$

$$= -\frac{iq}{8m^2}\,\alpha_i\alpha_j\left(p_i E_j - E_i p_j\right) + \text{h.o.t}$$

where we replace $\pi_i \to p_i$. Now use

$$\alpha_i\alpha_j = \delta_{ij} + i\,\varepsilon_{ijk}\Sigma_k .$$

From the δ_{ij} term we get

$$\frac{-iq}{8m^2}\left(\vec{p}\cdot\vec{E} - \vec{E}\cdot\vec{p}\right) = \left(\frac{-iq}{8m^2}\right)\left(-i\,\nabla\cdot\vec{E}\right)$$

$$= -\frac{q}{8m^2}\,\nabla\cdot\vec{E} = +\frac{q}{8m^2}\,\nabla^2\Phi .$$

Now do the other term:

$$\left(\frac{-iq}{8m^2}\right)\left(i\,\varepsilon_{ijk}\Sigma_k\right)\left(p_i E_j - E_i p_j\right).$$

$$-\frac{\partial^2\Phi}{\partial x_i\,\partial x_j}$$

Use $\quad p_i E_j = E_j p_i - i\left(\dfrac{\partial E_j}{\partial x_i}\right)$

The second term vanishes when contracted against ε_{ijk}, so this term becomes

$$\left(\frac{-iq}{8m^2}\right)\left(i\,\varepsilon_{ijk}\,\Sigma_k\right)\underbrace{\left(E_j P_i - E_i P_j\right)}_{\hookrightarrow\; -2\varepsilon_{ij\ell}(\vec{E}\times\vec{P})_\ell}$$

$$= \quad -\frac{q}{4m^2}\,\vec{\Sigma}\cdot(\vec{E}\times\vec{P}).$$

Altogether, ~~through~~ we have

$$H_4' = \quad -\frac{1}{8m^3}\beta\, p^4 \quad + \frac{q}{8m^2}\nabla^2\Phi \quad - \frac{q}{4m^2}\vec{\Sigma}\cdot(\vec{E}\times\vec{P}).$$

It is an even operator.

The new Hamiltonian still has odd terms at 3rd order,

$$\begin{array}{ccccccc} & & \eta^0 & & \eta^2 & \eta^3 & \eta^4 \\ H' & = & m\beta & + & H_2' & + H_3' & + H_4' \end{array}.$$

To eliminate these, we perform a 2nd F-W transformation,

$$\psi'' = e^{S'}\psi'$$

$$H'' = e^{S'}H'e^{-S'} \quad + \left(\frac{i\partial e^{S'}}{\partial t}\right)e^{-S'}.$$

We must choose ~~a~~ $S' \sim \eta^3$ to kill H_3'. Then the series is ~~$H'' = mβ$~~

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$$
\begin{array}{lcccccccccccc}
 & \eta^0 & & \eta^1 & & \eta^2 & & \eta^3 & & \eta^4 & & \\
H'' = & m\beta & + & 0 & + & 0 & + & L_{S'}(m\beta) & + & 0 & + & \cdots \\
 & & & & + & H_2' & + & 0 & + & 0 & + & \cdots \\
 & & & & & & + & H_3' & + & 0 & + & \cdots \\
 & & & & & & & & + & H_4' & + & \cdots
\end{array}
$$

There is a term $\frac{1}{2} L_{S'}^2 (m\beta)$, but it is order η^6, and a term $L_{S'} H_2'$, but it is order η^5. And $\dot{S}'$ is at least order η^5, since it is $\partial/\partial t$ applies to $S' \sim \eta^3$. So the table above is all we have.

We now choose S' such that

$$L_{S'}(m\beta) + H_3' = 0.$$

We don't have to solve this eqn. for S', it is not necessary. The result is

$$H'' = m\beta + H_2' + H_4'$$

all terms we have calculated already. Explicitly, this is

$$H'' = \beta\left[m + \frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\Sigma}\cdot\vec{B} - \frac{p^4}{8m^3}\right]$$

$$+ \frac{q}{8m^2}\nabla^2\Phi - \frac{q}{4m^2}\vec{\Sigma}\cdot(\vec{E}\times\vec{p}).$$

Restricting this to the upper 2-component spinor, we get a Pauli Hamiltonian,

$$H_{Pauli} = m + \frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}\cdot\vec{B} - \frac{p^4}{8m^3}$$

$$+ \frac{q}{8m^2}\nabla^2\Phi - \frac{q}{4m^2}\vec{\sigma}\cdot(\vec{E}\times\vec{p})$$

$\uparrow$ Pauli $\vec{\sigma}$

The term $-\frac{p^4}{8m^3} \to -\frac{p^4}{8m^3c^2}$ in ordinary units, is the relativistic kinetic energy correction.

The term $+\frac{q}{8m^2}\nabla^2\Phi \to +\frac{q}{8}\left(\frac{\hbar}{mc}\right)^2\nabla^2\Phi$ in ordinary units, is the Darwin term. To compare this to the presentation in Notes 23, we set $q = -e$ and $\Phi = \frac{Ze}{r}$ for an H-like atom, so $\nabla^2\Phi = -4\pi Z\delta^3(\vec{r})$. Then this term becomes

$$Ze^2\frac{\pi}{2}\frac{\hbar^2}{m^2c^2}\delta^3(\vec{r})$$

see Eq. (23.5a).

The last term is the spin-orbit term. Set $q = -e$, $\vec{E} = -\nabla\Phi$, $q\Phi = V = V(r)$, so

$$q\vec{E} = -q\nabla\Phi = -\frac{q}{r}\frac{d\Phi}{dr}\vec{r} = -\frac{1}{r}\frac{dV}{dr}\vec{r}.$$

Then the term becomes

$$\frac{1}{4m^2}\frac{1}{r}\frac{dV}{dr}(\vec{r}\times\vec{p})\cdot\vec{\sigma} \qquad \text{(natural units)}$$

$$\to \frac{1}{4}\left(\frac{\hbar}{mc}\right)^2\frac{1}{r}\frac{dV}{dr}\frac{(\vec{r}\times\vec{p})\cdot\vec{\sigma}}{\hbar} \qquad \text{or with } \begin{array}{l}\vec{L} = \vec{r}\times\vec{p}\\ \vec{S} = \frac{\hbar}{2}\vec{\sigma}\end{array}$$

it becomes

$$\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{L}\cdot\vec{S}.$$

see Eq. (25.12). The extra factor of 1/2 due to Thomas precession is automatically incorporated.