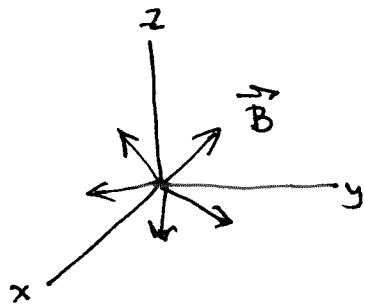


①

Magnetic monopole. A magnetic monopole is a hypothetical particle that has a Coulomb-like magnetic field,

$$\vec{B} = \mu \frac{\hat{r}}{r^2},$$

where μ is the strength of the monopole which is placed at the origin of a coordinate system.



Following Dirac, we investigate the motion of an electron ($q = -e$) in the monopole field. For simplicity we assume the monopole is infinitely massive, that it is at $r=0$, and we ignore the electron spin.

We need a vector potential such that $\vec{B} = \nabla \times \vec{A}$.

By writing out the curl in spherical coordinates, it is easy to find a solution. Of course the solution is not unique since we can always do a gauge transformation, but one possibility is

$$\vec{A} = \mu \frac{(1-\cos\theta)}{r \sin\theta} \hat{\phi}.$$

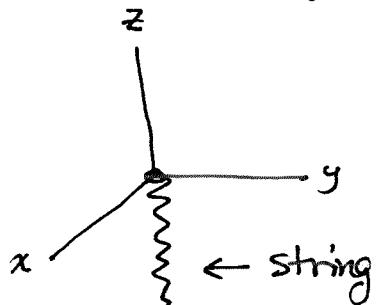
\vec{A} is purely in the ϕ -direction. Because of the $\sin\theta$ in the denominator, it looks like there may be a singularity when $\theta = 0$ or $\theta = \pi$. But $\theta = 0$ is not singular, because $1 - \cos\theta \rightarrow 0$ when $\theta \rightarrow 0$, in fact

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos\theta}{\sin\theta} = 0.$$

But $\theta = \pi$ is a singularity, since $1 - \cos\theta \rightarrow 2$ as $\theta \rightarrow \pi$.

The vector potential is singular on the negative z-axis (where $\theta = \pi$). Of course we expect \vec{A} to be singular when $r=0$, because \vec{B} is singular there, but we find that \vec{A} is singular on the entire negative z-axis, where \vec{B} is well behaved (as long as $z \neq 0$). There is nothing special about the negative z-axis as far as \vec{B} is concerned.

The negative z-axis (wavy line) is called the string of the monopole. It is really a misnomer; it should be called the string of the vector potential.



By doing a gauge transformation, we can move the string around.

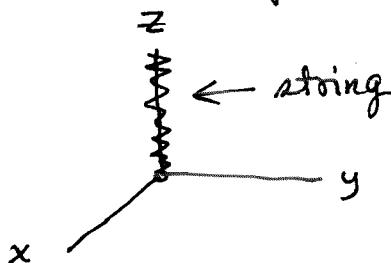
For example, in spherical coordinates,

$$\nabla\phi = \frac{1}{r\sin\theta} \hat{\phi}$$

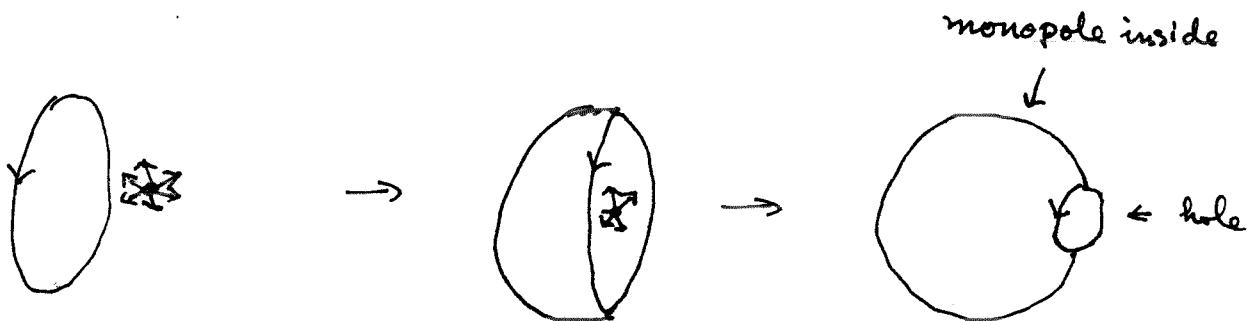
(the gradient of the azimuthal angle ϕ) so if we subtract $-2\mu\nabla\phi$ from the vector potential above, we get a new one,

$$\vec{A} = -\frac{\mu(1+\cos\theta)}{r\sin\theta} \hat{\phi}.$$

This is singular on the positive z-axis, but regular elsewhere:



Can we get rid of the string by means of some gauge transformation? That is, does there exist some smooth \vec{A} such that $\vec{B} = \nabla \times \vec{A}$, where \vec{B} is the monopole field? (We mean, \vec{A} is smooth except at $r=0$.) The answer is no, as we prove by contradiction. Suppose such a smooth \vec{A} exists, for $r>0$. Take a disc near the monopole and deform it to make a sphere with a hole in it, surrounding the monopole:



Now integrate \vec{A} around the edge of the disk, which turns into the edge of the hole in the sphere. By Stokes' theorem, this is the magnetic flux through the surface.

$$\oint \vec{A} \cdot d\vec{l} = \int_{\text{surface}} \vec{B} \cdot d\vec{a} = \text{flux.}$$

As we let the hole in the sphere shrink to 0, $\oint \vec{A} \cdot d\vec{l} \rightarrow 0$ since \vec{A} is smooth (we assume). But the flux $\rightarrow 4\pi \mu_0$, by Gauss' law. So there is a contradiction. We see that strings of \vec{A} are inevitable.

Let us write the two gauges we have discovered as

$$\vec{A}_N = \mu \frac{(1-\cos\theta)}{r \sin\theta} \hat{\phi}$$

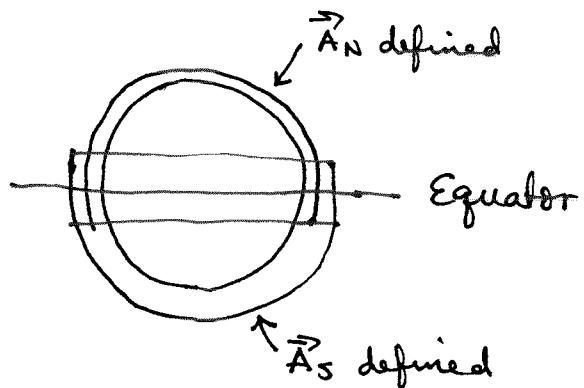
$$\vec{A}_S = -\mu \frac{(1+\cos\theta)}{r \sin\theta} \hat{\phi}$$

where

$$\vec{A}_N = \vec{A}_S + 2\mu \nabla \phi.$$

\vec{A}_N is regular in the northern hemisphere ($0 \leq \theta \leq \pi/2$) and can be extended below the equator ($\theta > \pi/2$) into the southern hemisphere; while \vec{A}_S is regular in the southern hemisphere ($\pi/2 \leq \theta \leq \pi$) and can be extended above the equator ($\theta < \pi/2$) into the northern hemisphere. In this way we cover the region $r > 0$ with two patches and two vector potentials, where \vec{A} is well behaved and nonsingular in each patch.

The two patches overlap on a strip around the equator, where \vec{A}_N and \vec{A}_S are connected by the gauge transformation above.



Therefore the wave function of the electron is also defined in two ways on the two patches.

Call the wave functions ψ_N and ψ_S on the two patches. Since

$$\text{Since } (\vec{p} + \frac{e}{c} \vec{A}_N) \psi_N = (-i\hbar \nabla + \frac{e}{c} \vec{A}_S + \frac{2e\mu}{c} \nabla \phi) \psi_N$$

the gauge transformation on the wave function must be

$$\psi_N = \psi_S e^{-\frac{2ie\mu}{\hbar c}\phi}$$

so that

$$(\vec{p} + \frac{e}{c} \vec{A}_N) \psi_N = e^{-\frac{2ie\mu}{\hbar c}\phi} (\vec{p} - \frac{e}{c} \vec{A}_S) \psi_S.$$

But both ψ_N and ψ_S must be single-valued on their respective domains of definition. Since ϕ goes from 0 to 2π around the equator, we must have

$$\frac{2e\mu}{\hbar c} = n = \text{integer.}$$

This implies

$$e = \frac{n\hbar c}{2\mu}.$$

Thus if even one monopole exists in the universe, it provides an explanation of the quantization of electric charge.