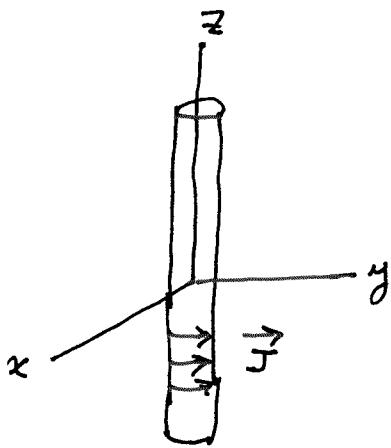
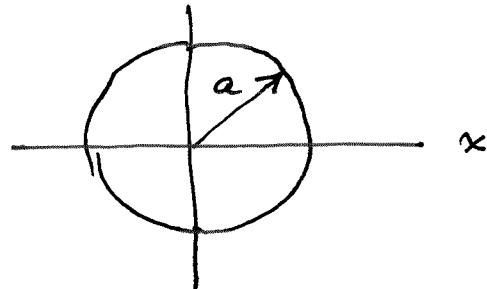


The Aharonov-Bohm effect concerns the behavior of a charged particle in the field of a long solenoid. The solenoid is centered on the z -axis and extends to $\pm\infty$. Its radius is a .



From above:



The surface of the solenoid is a current sheet that gives rise to a uniform magnetic field $\vec{B} = B \hat{z}$ inside the solenoid, and $\vec{B} = 0$ outside.

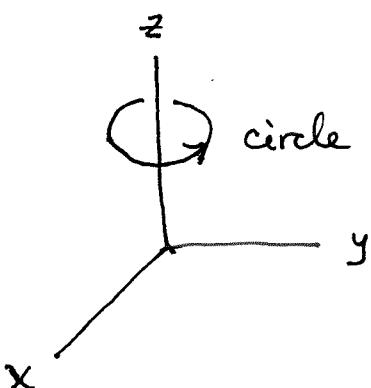
$$\vec{B} = \begin{cases} B \hat{z}, & p < a \\ 0, & p > a \end{cases}$$

We use cylindrical coordinates, where $p = \sqrt{x^2 + y^2}$.

Cyl. coords = (p, ϕ, z) .

For quantum mechanics we need a vector potential.

Assume a vector potential purely in the ϕ -direction, $\vec{A} = A_\phi \hat{\phi}$. We can find A_ϕ by integrating \vec{A} around a circle centered on the solenoid. The flux intercepted by



the circle depends on whether the radius of the circle p is $< a$ or $> a$.

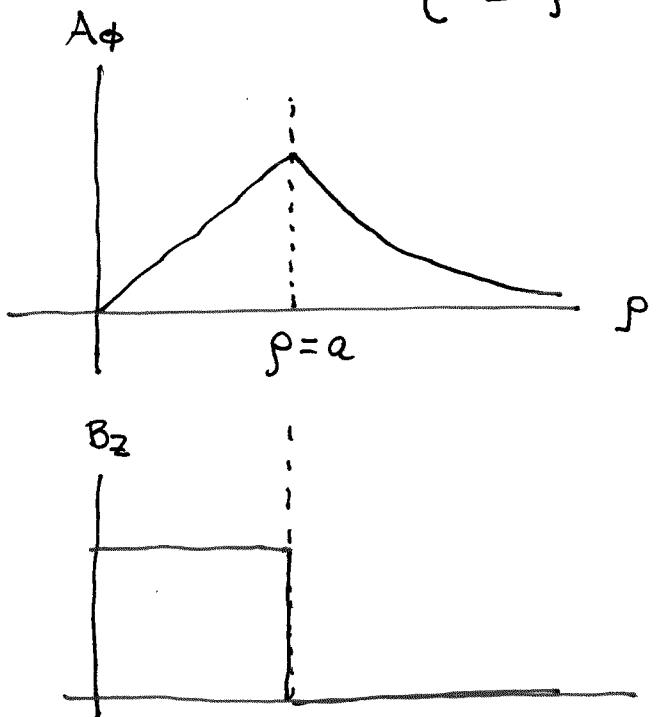
$$\oint \vec{A} \cdot d\vec{l} = 2\pi p A_\phi = \int_{\text{surf.}} \vec{B} \cdot d\vec{a} = \text{flux}$$

$$= \begin{cases} \pi p^2 B, & p < a \\ \pi a^2 B, & p > a \end{cases}$$

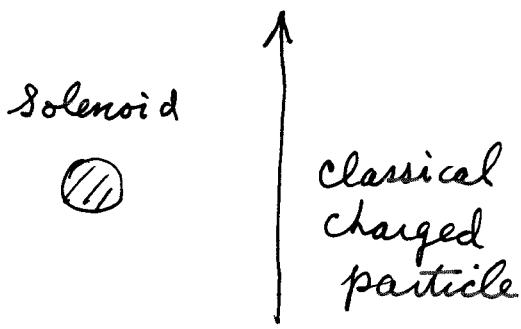
(2)

So,

$$A_\phi = \begin{cases} \frac{B}{2} \rho, & \rho < a \\ \frac{B}{2} \frac{a^2}{\rho}, & \rho > a \end{cases}$$



The magnetic field is 0 outside the solenoid, but the vector potential is not. A classical particle of charge $q = e$ in the exterior region is free; it feels no force since $\vec{B} = 0$ there.



But \vec{A} appears in the Schrödinger equation. Does this mean that \vec{A} has an effect on a quantum charged particle? Can the particle be used to measure \vec{A} ?

Let the particle have charge $q = e$, and let the solenoid be surrounded by a hard wall, so ψ satisfies the boundary conditions $\psi = 0$ at $\rho = a$. There may be other boundary conditions, for example, at ∞ .

Let $\psi_A(x, y, z) = \psi_A(\rho, \phi, z)$ be a solution of the Schrödinger equation,

$$\frac{1}{2m} \left(\vec{p} - \frac{e}{c} \vec{A} \right)^2 \psi_A = \begin{cases} i\hbar \frac{\partial \psi}{\partial t} \\ \text{or } E \psi \end{cases}$$

(3)

in the presence of the vector potential $\vec{A} = \frac{B}{2} \frac{a^2}{\rho} \hat{\phi}$, exterior region only. Is the solution ψ_A physically equivalent to a free particle solution? That is, can we convert ψ_A into a free particle solution by a gauge transformation?

This seems logical, since in the exterior region $\nabla \times \vec{A} = 0$, so theorems of calculus imply that $\vec{A} = \nabla g$ for a scalar field $g(x, y, z)$. Then using g in a gauge transformation we should be able to get a new gauge potential $\vec{A}' = \vec{A} - \nabla g = 0$.

As for the wave function, it should transform according to

$$\psi' = \psi_A e^{-ieg/\hbar c}$$

that is ψ' should be a solution of the free particle Schrödinger equation,

$$\frac{p^2}{2m} \psi' = \begin{cases} i\hbar \frac{\partial \psi'}{\partial t} \\ \text{or} \\ E \psi' \end{cases}$$

Let's see if this is true.

Let ϕ be the azimuthal angle (same notation as above).

Then in cyl. coords,

$$\nabla \phi = \frac{1}{\rho} \hat{\phi},$$

so if $\vec{A} = \frac{B}{2} \frac{a^2}{\rho} \hat{\phi}$, then

$$\vec{A} = \frac{Ba^2}{2} \nabla \phi, \quad g = \frac{Ba^2}{2} \phi.$$

Write this as

$$g = \frac{B\pi a^2}{2\pi} \phi = \frac{\Phi}{2\pi} \phi,$$

where Φ is the flux in the solenoid (not to be confused with the azimuthal angle ϕ). Thus the supposed free particle solution is

$$\begin{aligned}\psi' &= \psi_A e^{-i \frac{e\Phi}{2\pi\hbar c} \phi} \\ &= \psi_A e^{-i \left(\frac{\Phi}{\Phi_0}\right) \phi}\end{aligned}$$

where

$$\Phi_0 = \frac{2\pi\hbar c}{e} = \frac{\hbar c}{e} = \text{a flux quantum.}$$

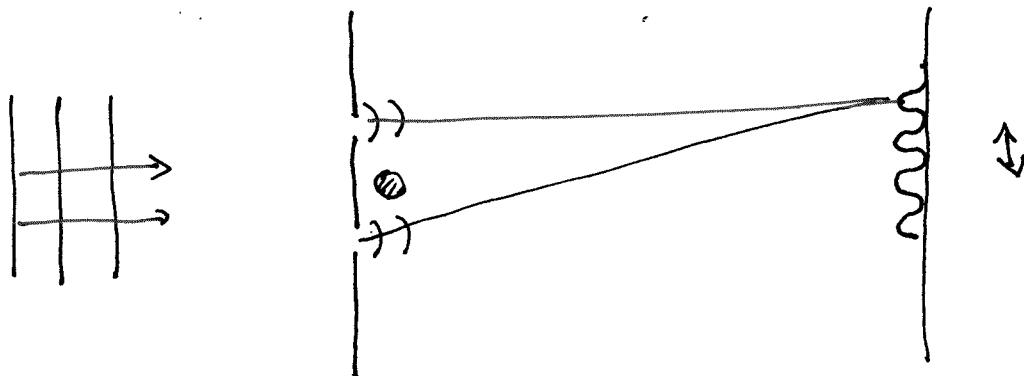
The problem with ψ' is that unless Φ is an integer multiple of a flux quantum, then ψ' suffers a discontinuity when ϕ goes from 0 to 2π . ψ_A must be a continuous function of (x, y, z) , since that is required if it is an acceptable solution of the Schrödinger equation. So unless $\Phi = n\Phi_0$, then ψ' is not an acceptable solution of the free particle Sch.-eqn. The solution in the presence of the solenoid is not physically equivalent to a free particle solution.

Conversely, an acceptable (continuous) solution of the free particle Schrödinger equation cannot be mapped into an acceptable sol'n of the Sch. eqn. with $\vec{A} = \frac{B}{2} \frac{a^2}{\rho} \hat{\phi}$ unless $\Phi = n\Phi_0$.

So what physical effects does the solenoid cause when $\Phi \neq n\Phi_0$? These must be nonclassical. (BTW, there is no requirement that the flux in the solenoid must be an integer multiple of a "flux quantum"; the name is a misnomer, since magnetic flux can take on any value).

Obviously the answer is, in a sense, that the wave function in the presence of the solenoid acquires an extra phase of $e^{2\pi i(\Phi/\Phi_0)}$ on going around the solenoid, as compared to a free particle solution.

One place where this has physical consequences is in a double slit experiment. In an ordinary double slit experiment, the peaks of the interference fringes on the screen are the places where the phase difference between the two paths from the two slits is an integer multiple of 2π .



But if there is flux in the solenoid behind the screen, there is an extra relative phase shift of $2\pi(\Phi/\Phi_0)$. Thus the interference fringes shift up and down, and the magnetic field in the solenoid has a nonlocal effect on the charged particle.

Notice that the observable effect depends only on Φ , that is, it is gauge invariant. Quantum physics is gauge-invariant just like classical physics.