Examples of Bohr–Sommerfeld quantization.

Usual rule:

\[
\text{area} = \oint p\,dx = (n+\frac{1}{2})(2\pi\hbar).
\]

Area of orbit in phase space. This comes from idea that as particle moves in its orbit, it accumulates a phase of \( \frac{1}{\hbar} \oint p\,dx \), but it loses \( \pi/2 \) phase when passing through turning points. So, if there are 2 turning points, then the total phase accumulated on a single orbit is

\[
\frac{1}{\hbar} \oint p\,dx - 2 \left( \frac{\pi}{2} \right) = 2n\pi
\]

which must be an integer multiple of \( 2\pi \), as indicated. This is the same as the B-S rule above.

(Example of harmonic oscillator done in notes.)

If the wall the particle bounces off is hard, then the phase shift is \( -\pi \) at each bounce, instead of \( -\pi/2 \). That's because \( e^{-i\pi} = -1 \), so the reflected wave cancels the incident wave at the wall, where boundary conditions require \( \Psi = 0 \).

For example, the particle in a box,

\[
V(x)
\]

2 hard walls, particle bounces back and forth with momentum \( p_0 \) (say) energy \( E = p_0^2/2m \).
Orbit in phase space:

\[ P \\
\text{Pu} \quad \rightarrow \quad \rightarrow \quad x \text{ (wall)} \quad \rightarrow \quad x=0 \]

Orbit is rectangle, area = \( 2P_0L \).

Quantization condition,

\[ \frac{1}{\hbar} \oint p \, dx - 2 \left( \frac{\pi}{\hbar} \right) = 2n\pi, \]

not \( \pi/2 \), 2 hard walls

or

\[ \text{area} = \oint p \, dx = (n+1) \left( 2\pi\hbar \right). \]

\[ = 2P_0L, \]

or

\[ P_0 = \frac{(n+1)\pi\hbar}{L}, \quad E = \frac{P_0^2}{2m} = \frac{(n+1)^2\pi^2\hbar^2}{2mL^2}. \]

The exact answer, with \( n = 0, 1, 2, \ldots \)

\[ n+1 = 1, 2, 3, \ldots \]
Another example, the rigid rotor: First we do it classically. Two masses \( m_1, m_2 \) are connected by a massless rod of length \( R \).

\[
\text{Reduced mass } \mu, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}
\]

Moment of inertia
\[
I = \mu R^2.
\]

This is a model of a diatomic molecule.

Rotor rotates about its center of mass in the \( x-y \) plane (let's say), with angle \( \varphi \):

The Lagrangian is just the kinetic energy:
\[
L = \frac{1}{2} I \dot{\varphi}^2
\]

The momentum conjugate to the angle \( \varphi \) is actually the angular momentum, or, rather, its \( z \)-component:
\[
\mathbf{p}_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = I \dot{\varphi} = L_z.
\]

The Hamiltonian is
\[
H = \mathbf{p}_\varphi \cdot \dot{\mathbf{r}} - L = \frac{L_z^2}{2I}.
\]
This is the classical Hamiltonian. What should the Schrödinger equation be? If we had a Cartesian coordinate $x$, then we can quantize a classical Hamiltonian $H(x,p)$ by making the replacements, $x \rightarrow$ multiplication by $x$, $p \rightarrow -i \frac{\hbar}{\partial x}$. These operators act on wave functions $\psi(x)$. This is called Dirac quantization. We guess we can do the same thing for an angular coordinate like $\phi$. Then $\phi \rightarrow$ multiplication by $\phi$, $L_z \rightarrow -i \hbar \frac{\partial}{\partial \phi}$ when acting on wave functions $\psi(\phi)$.

Also, since $\phi$ is a periodic coordinate, we require

$$\psi(\phi + 2\pi) = \psi(\phi).$$

Then if $H = \frac{\hbar^2}{2I}$, the Schrödinger equation is

$$-\frac{\hbar^2}{2I} \frac{d^2\psi}{d\phi^2} = E \psi(\phi).$$

The normalized solutions are

$$\psi_m(\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}}, \quad m=0, \pm 1, \pm 2, \ldots$$

with energies,

$$E_m = \frac{\hbar^2 m^2}{2I}.$$

These are also eigenfunctions of $L_z$, with eigenvalues

$$L_z = m\hbar.$$

To apply the B-S rules we must look at the orbits in phase space.
The orbits are straight lines at const. \( L_z \). They look something like the orbits of the particle in the box, but they don't bounce since there is no hard wall. There is no soft wall, either; the particle just continues forever with the same \( \varphi \). Since \( \varphi \) is periodic, the actual motion is a circle. The quantization condition must be

\[
\frac{1}{\hbar} \oint L_z \, d\varphi = 0 = 2m\pi
\]

\[\uparrow \text{no turning point,}\]

or, \textit{Area under curve} = \( 2m\pi \hbar \). This is

\[2\pi L_z = 2m\pi \hbar,\]

or

\[L_z = m\hbar,\]

\[E = \frac{L_z^2}{2I} = \frac{m^2\hbar^2}{2I}.\]

The exact answers.
A Planck cell has an area of $2\pi \hbar$ in one-dimensional systems. In higher dimensions it has a volume of $(2\pi \hbar)^n$ in phase space, where $n$ is the number of coordinate in configuration space. The dimension of phase space is $2n$.

Consider, for example, a free particle confined to some region $R$ of 3-dimensional space.

Let $N(E_0) =$ number of energy eigenstates with energy $\leq E_0$. Then by counting Planck cells we have the estimate,

$$N(E_0) = \frac{1}{(2\pi \hbar)^3} \int d^3x \int d^3p \quad H(x,p) \leq E_0.$$ 

The integral is over a 6-dimensional region of phase space, but it is easy because $H = \frac{1}{2m} \vec{p}^2$. Thus:

$$\int d^3x \int d^3p = \int_R d^3x \int_{\vec{p}} d^3p \quad \text{where} \quad p_0 = \sqrt{2mE_0}.$$ 

But

$$\int_R d^3x = V$$

$$\int d^3p = \frac{4}{3} \pi p_0^3 = \frac{4}{3} \pi (2mE_0)^{3/2}.$$
Thus,

\[
N(E) = \frac{V}{(2\pi \hbar)^3} \frac{4}{3} \pi (2mE)^{3/2}
\]

This is an asymptotic estimate for \( N(E) \). It does not take into account any phase shifts at turning points because when \( E \) (or \( N \)) is large these won’t matter. If the particle has spin we should multiply this by 25+1.

By differentiating we get the density of states:

\[
\frac{dN}{dE} = \frac{V}{(2\pi \hbar)^3} 4\pi m \sqrt{2mE}
\]

When people talk about the "density of states" they usually mean the number of energy eigenstates per unit energy interval. But if we ask for the number per unit volume in phase space, the answer is \( \frac{1}{(2\pi \hbar)^3} \), independent of the Hamiltonian.